Graph-theoretic Algorithm for Nonlinear Power Optimization Problems

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Joint work with Ramtin Madani, Somayeh Sojoudi, Ghazal Fazelnia
Optimization for Power Networks

- **Optimizations:**
  - Optimal power flow (OPF)
  - Security-constrained OPF
  - State estimation
  - Network reconfiguration
  - Unit commitment
  - Dynamic energy management

- **Issue of non-convexity:**
  - Discrete parameters
  - Nonlinearity in continuous variables

- **Transition from traditional grid to smart grid:**
  - More variables (10X)
  - Time constraints (100X)
OPF-based problems solved on different time scales:
- Electricity market
- Real-time operation
- Security assessment
- Transmission planning

Existing methods based on linearization or local search

**Question:** How to find the best solution using a scalable robust algorithm?

Huge literature since 1962 by power, OR and Econ people
Consider a polynomial optimization
\[
\min_{x \in \mathbb{R}^{n-1}} \ f_0(x)
\]
s.t. \( f_k(x) \leq 0 \) for \( k = 1, \ldots, p \)

With no loss of generality, assume that \( f_k \)'s are quadratic.

Consider the transformation:
\[
X \triangleq [1 \ x^T] [1 \ x^T]
\]

Rank-constrained reformulation:
\[
\min_{X \in \mathbb{S}^n} \ \text{trace} \{ F_0 X \}
\]
s.t. \( \text{trace} \{ F_k X \} \leq 0 \) for \( k = 1, \ldots, p \)
\( X_{11} = 1 \)
\( X \succeq 0 \)
\]
Review of four previous projects on SDP relaxation

New results:
- Notion of complexity for power networks (treewidth)
- Connection between rank and treewidth
- Case studies:
  1. **IEEE 300 bus**: Rank for security constrained OPF $\leq 7$
  2. **Polish 3120 bus**: Rank for security constrained OPF $\leq 27$
Projects 1-3

Project 1: How to solve a given OPF in polynomial time? (joint work with Steven Low)

- SDP relaxation works for various systems (IEEE systems, etc.).
- This is due to passivity.

Project 2: Find network topologies over which optimization is easy? (joint work with Somayeh Sojoudi, David Tse and Baosen Zhang)

- Distribution (acyclic) networks are fine (under certain assumptions).
- Transmission networks may need phase shifters.

Project 3: How to design a distributed algorithm for solving OPF? (joint work with Stephen Boyd, Eric Chu and Matt Kranning)

- A practical (infinitely) parallelizable algorithm using ADMM.
- It solves 10,000-bus OPF in 0.85 seconds on a single core machine.
Project 4: How to do optimization for mesh networks? (joint work with Ramtin Madani and Somayeh Sojoudi)

- Penalized SDP relaxation:

\[
\sum_{k \in G} f_k(P_{G_k}) \quad \rightarrow \quad \sum_{k \in G} f_k(P_{G_k}) + \varepsilon \sum_{k \in G} Q_{G_k}
\]

- Example borrowed from Bukhsh et al.:
  - Modify IEEE 118-bus system has 3 local solutions with the optimal costs 129625.03, 177984.32 and 195695.54.
  - Our method finds the best one.
Response of SDP to Equivalent Formulations

- **Capacity constraint:** active power, apparent power, angle difference, voltage difference, current?

![Diagram of power flow and equations](image)

1. Equivalent formulations behave differently after relaxation.
2. Problem D has an exact relaxation.
Treewidth

- **Tree decomposition:**
  
  We map a given graph $G$ into a tree $T$ such that:
  
  - Each node of $T$ is a collection of vertices of $G$
  - Each edge of $G$ appears in one node of $T$
  - If a vertex shows up in multiple nodes of $T$, those nodes should form a subtree

  - **Width of a tree decomposition:** The cardinality of largest node minus one

  - **Treewidth of graph:** The smallest width of all tree decompositions
- Treewidth of a tree: 1

- How about the treewidth of IEEE 14-bus system with multiple cycles? 2

- How to compute the treewidth of a large graph?
  - NP-hard problem
  - We used graph reduction techniques for sparse power networks
Power Networks

Upper bound on the treewidth of sample power networks:

<table>
<thead>
<tr>
<th>System $\mathcal{G}$</th>
<th>$\text{tw}{\mathcal{G}}$</th>
<th>System $\mathcal{G}$</th>
<th>Bound on $\text{tw}{\mathcal{G}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEEE 14-bus</td>
<td>2</td>
<td>Polish 2383-bus</td>
<td>26</td>
</tr>
<tr>
<td>IEEE 30-bus</td>
<td>3</td>
<td>Polish 2736-bus</td>
<td>38</td>
</tr>
<tr>
<td>New England 39-bus</td>
<td>3</td>
<td>Polish 2746-bus</td>
<td>40</td>
</tr>
<tr>
<td>IEEE 57-bus</td>
<td>5</td>
<td>Polish 3012-bus</td>
<td>28</td>
</tr>
<tr>
<td>IEEE 118-bus</td>
<td>4</td>
<td>Polish 3120-bus</td>
<td>26</td>
</tr>
<tr>
<td>IEEE 300-bus</td>
<td>6</td>
<td>Polish 3375-bus</td>
<td>28</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\min & \quad \text{trace}\{F_0 X\} \\
\text{s.t.} & \quad \text{trace}\{F_k X\} \leq 0 \quad \text{for} \quad k = 1, \ldots, p \\
& \quad X_{11} = 1 \\
& \quad X \succeq 0
\end{align*}
\]

- Define $G$ as the sparsity graph
- **Theorem**: There exists a solution with rank at most treewidth of $G + 1$
- We proposed infinitely many optimizations to find that solution.
Example: Consider the security-constrained unit-commitment OPF problem.

- Use SDP relaxation for this mixed-integer nonlinear program.

- What is the rank of $X^{opt}$?
  
  1. **IEEE 300-bus system**: rank ≤ 7
  
  2. **Polish 3120-bus system**: Rank ≤ 27

How to go from low-rank to rank-1? Penalization (tested on 7000 examples)

- IEEE 14-bus system
- IEEE 30-bus system
- IEEE 57-bus system
Polynomial Optimization

- Sparsification Technique: distributed computation

\[ x_i \iff (x_{i1}, x_{i2}) \text{ s.t. } x_{i1} = x_{i2} \]

- This gives rise to a sparse QCQP with a sparse graph.

- The treewidth can be reduced to 2.

**Theorem:** Every polynomial optimization has a QCQP formulation whose SDP relaxation has a solution with rank 1, 2 or 3.
Distributed Control

- Computational challenges arising in the control of real-world systems:
  - Communication networks
  - Electrical power systems
  - Aerospace systems
  - Large-space flexible structures
  - Traffic systems
  - Wireless sensor networks
  - Various multi-agent systems

Decentralized control

Distributed control
Optimal Decentralized Control Problem

- **Optimal centralized control:** Easy (LQR, LQG, etc.)
- **Optimal distributed control (ODC):** NP-hard (Witsenhausen’s example)

Consider the time-varying system:

\[
\begin{align*}
    x[\tau + 1] &= A[\tau] x[\tau] + B[\tau] u[\tau], \\
y[\tau] &= C[\tau] x[\tau]
\end{align*}
\quad \forall \tau \in \mathbb{Z}_+
\]

The goal is to design a structured controller \( u[\tau] = K y[\tau] \) to minimize

\[
\sum_{\tau=0}^{p} \left( x[\tau]^T Q[\tau] x[\tau] + u[\tau]^T R[\tau] u[\tau] \right) + \mu \text{ trace}\{KK^T\}
\]
Two Quadratic Formulations in Static Case

- **Formulation in time domain:**
  - Stack the free parameters of $K$ in a vector $h$.
  - Define $\mathbf{v}$ as:
    \[
    \mathbf{v} = \begin{bmatrix} 1 & h^* & x[0]^* & x[1]^* & \cdots & x[p]^* & y[0]^* & \cdots & y[p]^* & u[0]^* & \cdots & u[p]^* \end{bmatrix}^*
    \]

- **Formulation in Lypunov domain:**
  - Consider the BMI constraint:
    \[
    \begin{bmatrix}
    P & P(A + BKC)^* \\
    (A + BKC)P & P
    \end{bmatrix} \succeq 0
    \]
  - Define $\mathbf{v}$ as:
    \[
    \mathbf{v} = \begin{bmatrix} 1 & h^* & P_{11} & P_{12} & \cdots & P_{1n} & \cdots & P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}^*
    \]
Theorem: The SDP relaxation of ODC has a solution with rank 1-3 in the static/dynamic case for the time-domain or Lyapunov-domain formulation.
Conclusions

- **Optimization over power networks:**
  - Complexity is related to treewidth.

- **Optimal decentralized control:**
  - NP-hard problem with a small treewidth.

- **General theory for polynomial optimization:**
  - Every polynomial optimization has an SDP relaxation with a rank 1-3 solution.

**References:**