Vector Quantization and Subband Coding

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Vector Quantization
Vector Quantization (VQ)

- Each image block has \( N \) pels
- Consider each image block as an \( N \)-D vector \( \mathbf{x} \)

\[
R_1 \quad y_1 \quad R_3 \quad R_2
\]

- Quantization: \( \mathbf{x} \to y_k \) if \( \mathbf{x} \in R_k \)
- \( y_k \): codewords or code vectors
- The set of \( y_k \) is called a codebook

Rate and Distortion

- If the number of codewords is \( K \), then the number of bits required to send one vector is \( \log_2 K \)
- Rate \( R \)
  - \( R \) bits per pixel
  - \( NR \) bits for one vector, so \( \log_2 K = NR \), i.e., \( \log K = 2^{NR} \)
- Distortion \( D \)
  - Given the probability density function \( p(\mathbf{x}) \) and distortion measure \( d(\mathbf{x}, \mathbf{y}) \), the average distortion is

\[
D = \sum_{k=1}^{K} \int_{R_k} d(\mathbf{x}, y_k)p(\mathbf{x})d\mathbf{x}
\]
• Given $y_k$, $R_k$ should be chosen such that

$$ R_k = \{ x : d(x, y_k) \leq d(x, y_j) \forall j \neq k \} $$

= the set of $x$ for which $y_k$ is the nearest point

• For $L_2$ norm, i.e.,

$$ d(x, y) = \frac{1}{N} \sum_{i=1}^{N} (x_i - y_i)^2 $$

we get

$$ d(x, y_k) = \frac{1}{N} \sum_{i=1}^{N} (x_i - y_k)^2 $$

Convex Polytope
(Voroni Cell)

Q: How about $L_1$ or $L_{\infty}$?

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• Given $R_k$, $y_k$ should be chosen such that

$$ \int_{R_k} d(x, y_k)p(x)dx \text{ is minimum} $$

• With $L_2$ norm, we get

$$ y_k = \text{centroid of } R_k = \int_{R_k} x p(x)dx $$

• In the discrete case, optimal $y_k$ is the average of the vectors in $R_k$
Generalized Lloyd Algorithm
(LBG Algorithm, K-means Algorithm)

• Linde, Buzo, and Gray, 1980

• Given \( p(\mathbf{x}) \), or given a set of training vectors
  (1) Start with an initial set of \( \mathbf{y}_k \), i.e., initial codebook
  (2) With the current \( \mathbf{y}_k \), calculate the region \( R_k \)
  (3) Replace each \( \mathbf{y}_k \) with the centroid of \( R_k \)
  (4) If the overall distortion \( D \) is lower than a threshold, stop.
    Otherwise, go to (2)

• Only gives local optimum. Proper choice of initial codebook is important

• Choice of initial codebook
  – A representative subset of the training vectors
  – Scalar quantization in each dimension
  – Splitting…

• Nearest Neighbor (NN) algorithm [Equitz, 1984]
  – Start with the entire training set
  – Merge the two vectors that are closest into one vector equal to their mean
  – Repeat until the desired number of vectors is reached, or the distortion exceeds a certain threshold
Properties of VQ

- Codebook design is very complex
  - 4x4 blocks at 1 bpp: \(2^{16}\) codewords
  - 16 images of size 256x256: \(2^{16}\) training vectors (4x4 each)
  - Codebook size: \(2^{16} \times 4 \times 4 \times 8\) bits = 8.3 Mbits

- More useful for low bitrate
  - 4x4 blocks at 0.5 bpp: \(2^8 = 256\) codewords
  - One 256x256 image: 4096 training vectors
  - Codebook size: \(256 \times 4 \times 4 \times 8\) bits = 32.8 Kbits

- Simple decoder, complex encoder
  - Very good for image retrieval

- Poor performance on images not in the training set vs. overhead of sending the codebook
VQ Variants and Improvements

- Multistage VQ
- Product Codes
  - Send mean and variance separately
- Classified VQ
  - Edges, texture areas, flat areas
- Predictive VQ
- VQ for color images
  - Exploit correlation among color components, e.g. R,G,B
  - YUV components are practically uncorrelated

Subband Coding
Subband Coding

- Decompose the signal in the frequency domain
- Critical downsampling (maximal decimation) maintains the number of samples in the subbands
- Wavelet coding: Recursively apply subband decomposition to the low freq band
- 2-D: Separable filtering to get 4 bands: LL, LH, HL, HH

Subband Coding vs. Transform Coding

Polyphase Representation
• Perfect reconstruction (PR) is obtained when $R(z) = E^{-1}(z)$
• When $E(z)$ and $R(z)$ are constant matrices, subband coding degenerated to blocked-based operation, i.e., transform coding
• In particular, if $E(z)$ is a DCT matrix and $R(z)$ is IDCT, this becomes DCT coding
• Subband coding can be viewed as transform coding with overlapped blocks. So, it can exploit correlation of pixels at longer range
• Coding Artifacts:
  – Transform Coding: blocking
  – Subband Coding: ringing, contouring

**Optimal Bit Allocation**

• We can allocate different bit rates to the subbands based on their properties
• Assume that we apply scalar quantization with bitrate $b_k$ to the subbands $x_k$, then the quantization error is

\[ \sigma_{q_k}^2 = c \times 2^{-2b_k} \sigma_{x_k}^2 \]

• The overall quantization error is

\[ \sigma_q^2 = \frac{1}{M} \sum_{k=1}^{M} \sigma_{q_k}^2 \]

• The overall bitrate is

\[ b = \frac{1}{M} \sum_{k=1}^{M} b_k \]
\[ \sigma_q^2 \geq \left( \prod_{k=1}^{M} \sigma_{q_k}^2 \right)^{\frac{1}{M}} \] (AM-GM inequality)

\[ = c \left( \prod_{k=1}^{M} 2^{-2b_k} \sigma_{s_k}^2 \right)^{\frac{1}{M}} = c \left( 2^{-2\sum b_k} \prod_{k=1}^{M} \sigma_{s_k}^2 \right)^{\frac{1}{M}} \]

\[ = c \times 2^{-2b} \left( \prod_{k=1}^{M} \sigma_{s_k}^2 \right)^{\frac{1}{M}} \] (a constant for given signal and filter bank)

- Equality holds if and only if \( \sigma_{q_k}^2 = \sigma_q^2 \quad \forall k \)
- Optimal bit allocation \( b_k = \frac{1}{2} \log \frac{c \times \sigma_{s_k}^2}{\sigma_q^2} \)
- Gain \( \frac{1}{M} \sum_{k=1}^{M} \sigma_{s_k}^2 \left( \prod_{k=1}^{M} \sigma_{s_k}^2 \right)^{\frac{1}{M}} \geq 1 \) No gain if \( \sigma_{s_k}^2 \) are identical

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**Pyramid Coding**

- Diagram showing the pyramid coding structure with levels and bit allocations.

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The “Pyramid”

- Consider the 2-D case

- Number of samples is 33% more

\[ N + \frac{N}{4} + \frac{N}{16} + \cdots \approx \frac{4}{3} N \]

- Non-critical sampling
- PR is always possible
  - No matter how L and Int are designed
- Progressive transmission is possible
## References

- **VQ**
- **Subband**
  - P.P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice Hall, 1993