Partially Ordered Set

A partially-ordered set (poset) $(S, \leq)$ is a set S combined with a binary relation $\leq$ indicating that 1 element comes before another. $\leq$ must satisfy:

1) Reflexive: $a \leq a$
2) Antisymmetric: if $a \leq b$ and $b \leq a$ then $a = b$
3) Transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$

* For a pair (a, b), if $a \leq b$ or $b \leq a$, then a & b are comparable

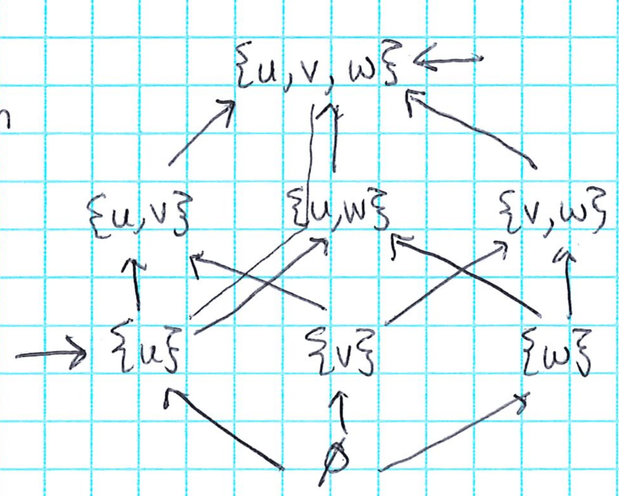
If all pairs are comparable, we have a totally ordered set. Otherwise partially ordered.

Example: Power Set $T = \{u, v, w\}$

$$S = \mathcal{P}(T) = \{\emptyset, \{u\}, \{v\}, \{w\}, \{u,v\}, \{u,w\}, \{v,w\}, \{u,v,w\}\}$$

$a, b \in S$, $a \leq b$ iff $a \subseteq b$.

Hasse Diagram



Ex Let $S = \{a, b, c\}$ $\quad a \leq b,\ b \leq c,\ c \leq a$

$a \neq b \neq c$

Is this a poset? No — transitivity is violated.

Complete Lattices

A **complete lattice** is a poset where all subsets have a supremum (join) and an infimum (meet).

Supremum ← smallest upper bound

Infimum ← greatest lower bound

We write $(S, \leq, \wedge, \vee, \perp, \top)$

$\perp$ - bottom

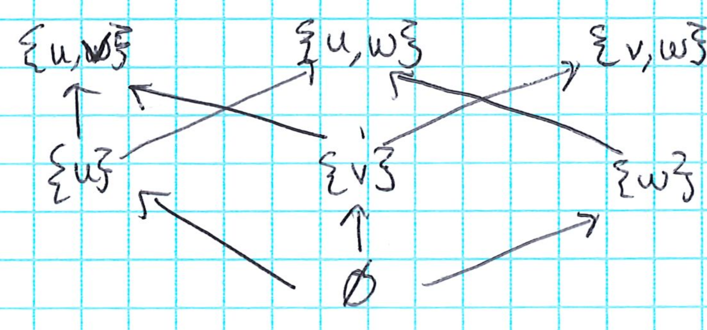
$\top$ - top

Example: Think back to power set poset. $(\mathcal{P}(T), \subseteq)$

$$\{u\} \vee \{w\} = \{u, w\} \qquad \perp = \emptyset$$

$$\bigvee \{\{u\},\{v\},\{w\}\} = \{u, v, w\} \qquad \top = \{u, v, w\}$$

Ex Define a poset that is not a complete lattice.



Key Question: How do we decide to deny/allow access?

Define both policies and requests as arrays of sets

Request: $T^G = [\{IP, Name\}, \{AdsData\}, \{Ads\}, \{\perp\}]$

$\uparrow$ Datatype $\quad \uparrow$ InStore $\quad \uparrow$ UseForPurpose $\quad \uparrow$ AccessByRole

Policy Clause $T^C = [\{\ \}, \{\ \}, \{\ \}, \{\ \}]$

Allow Define partial order $\sqsubseteq$ over vectors:

$T^G \sqsubseteq T^C$ iff $\forall$ attributes x, $T^G.x \sqsubseteq_x T^C.x$ where

$\sqsubseteq_x$ is defined as: $T_x \sqsubseteq T'_x$ iff $\forall_{v \in T_x} \exists v' \in T'_x : v \leq v'$

Allow if $T^G \sqsubseteq T^C$.

Deny Define $\sqcap$ as follows: Pointwise, where

$$T_x \sqcap T'_x = \{ \bigvee_{v \in T_x} v \wedge v', \forall v' \in T'_x \}$$

i.e. find largest lower bound $\forall$ pairs from 2 sets.

Ex: $T^G = [\underset{\text{Datatype}}{\{IP\}}, \underset{\text{Purpose}}{\{\top\}}]$ $\quad T^C = [\{Name\}, \{\top\}]$

$T^G \sqcap T^C = [\{\perp\}, \{\top\}]$

Is there at least 1 $\perp$ in $T^G \sqcap T^C$?

Yes → Allow

No → Deny