Introduction

- Last week we discussed the Z-transform at length, including the unit sample response, ROC, inverse Z-transforms and comparison to the DTFT and difference equations.

- Today we will discuss the frequency response of LSI systems and how it relates to the system function in Z-transform form.

- Specifically we will
  - Relate magnitude and phase of DTFT to locations of poles and zeros in z-plane.
  - Discuss several important special cases:
    - All-pass systems
    - Minimum/maximum-phase systems
    - Linear phase systems
Review - Difference equations and Z-transforms characterizing LSI systems

Many LSI systems are characterized by difference equations of the form

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]

They produce system functions of the form

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \]

Comment: This notation is a little different from last week's (but consistent with the text in Chap, 5)

Difference equations and Z-transforms characterizing LSI systems (cont.)

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \]

Comments:

—LSI systems characterized by difference equations produce z-transforms that are ratios of polynomials in z or z^{-1}.

—The zeros are the values of z that cause the numerator to equal zero, and the poles are the values of z that cause the denominator polynomial to equal zero.
Discrete-time Fourier transforms and the Z-transform

Recall that the DTFT is obtained by evaluating the z-transform along the contour \( z = e^{j\omega} \)

\[ H(e^{j\omega}) = H(z) \bigg|_{z = e^{j\omega}} \]

The DTFT is generally complex and typically characterized by its magnitude and phase:

\[ H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{j\omega k}} = |H(e^{j\omega})| e^{j\phi H(e^{j\omega})} \]

Obtaining the magnitude and phase of the DTFT by factoring the z-transform

Factoring the z-transform:

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left( \frac{b_0}{a_0} \right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \]

Comment: The constants \( c_k \) and \( d_k \) are the zeros and poles of the system respectively
So what do those terms mean, anyway?

Convert into a polynomial in \( z \) by multiplying numerator and denominator by largest power of \( z \):

\[
H(z) = \left( \frac{b_0}{a_0} \right) \frac{\prod_{k=1}^{M} (1-c_k z^{-1})}{\prod_{k=1}^{N} (1-d_k z^{-1})} = z^{N-M} \left( \frac{b_0}{a_0} \right) \frac{\prod_{k=1}^{M} (z-c_k)}{\prod_{k=1}^{N} (z-d_k)}
\]

Now consider one of the numerator terms, \((z-c_k)\).

Note that the vector \((z-c_k)\) is the length of line from the zero to the current value of \( z \) or the distance from the zero to the unit circle.

Finding the magnitude of the DTFT

\[
H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = z^{N-M} \left( \frac{b_0}{a_0} \right) \frac{\prod_{k=1}^{M} (z-c_k)}{\prod_{k=1}^{N} (z-d_k)}|_{z=e^{j\omega}}
\]

**Magnitude:**

\[
|H(e^{j\omega})| = |H(z)||_{z=e^{j\omega}} = \left| z^{N-M} \left( \frac{b_0}{a_0} \right) \frac{\prod_{k=1}^{M} (z-c_k)}{\prod_{k=1}^{N} (z-d_k)} \right|_{z=e^{j\omega}} = \left| \frac{b_0}{a_0} \right| \left| \prod_{k=1}^{M} (z-c_k) \right| \left| \prod_{k=1}^{N} (z-d_k) \right|_{z=e^{j\omega}}
\]

**Comment:** The magnitude is the product of magnitudes from zeros divided by product of magnitudes from poles.
Finding the phase of the DTFT

\[ H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = e^{N-M} \left( \frac{b_0}{a_0} \right) \prod_{k=1}^{M} (z-c_k) \prod_{k=1}^{N} (z-d_k) \]

n Phase:

\[ \angle H(e^{j\omega})|_{z=e^{j\omega}} = \omega (N-M) + \left( \frac{b_0}{a_0} \right) \sum_{k=1}^{M} \angle(z-c_k) - \sum_{k=1}^{N} \angle(z-d_k) \]

n Comment: The magnitude is the sum of the angles from the zeros minus the sums of the angles from the poles

Example 1: Unit time delay

\[ H(z) = z^{-1} \]

n Pole-zero pattern:

Frequency response:
Example 2: Decaying exponential sample response

\[ H(z) = \frac{1}{1 - \alpha z^{-1}} \]

- Pole-zero pattern:
- Frequency response:

Example 3: Notch filter

\[ H(z) = \frac{(z - e^{j\pi/4})(z - e^{-j\pi/4})}{(z - 0.95e^{j\pi/4})(z - 0.95e^{-j\pi/4})} \]

- Pole-zero pattern:
- Frequency response:
Summary (first half)

- The DTFT is obtained by evaluating the \( z \)-transform along the unit circle
- As we walk along the unit circle,
  - The magnitude of the DTFT is proportional of the product of the distances from the zeros divided by the product of the distances from the poles
  - The phase of the DTFT is (within additive constants) the sum of the angles from the zeros minus the sum of the angles from the poles
- After the break:
  - Allpass systems
  - Minimum-phase and maximum-phase systems
  - Linear-phase systems

Special types of LSI systems

- We can get additional insight about the frequency-response behavior of LSI systems by considering three special cases:
  - Allpass systems
  - Systems with minimum or maximum phase
  - Linear-phase systems
All-pass systems

Consider an LSI system with system function \( H(z) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \)
with \( \alpha \) complex.

Let \( \alpha = r e^{j\theta} \)

Then there is a pole at \( z = re^{j\theta} \)
And a zero at \( z = \frac{1}{r} e^{j\theta} \)

Comment: We refer to this configuration as mirror image poles and zeros.

Frequency response of all-pass systems

\( H(z) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \); \( z = e^{j\omega}, \alpha = re^{j\theta} \)

Obtaining magnitude of frequency response directly:

\[
|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{(e^{-j\omega} - re^{-j\theta})(e^{j\omega} - re^{j\theta})}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{j\omega})}
\]

\[
= \frac{(1-re^{j(\theta-\omega)} - re^{-j(\theta-\omega)} + r^2)}{(1-re^{-j(\theta-\omega)} - re^{j(\theta-\omega)} + r^2)} = 1
\]
Frequency response of all-pass systems

- All-pass systems have mirror-image sets of poles and zeros.

- All-pass systems have a frequency response with constant magnitude.

System functions with the same magnitude can have more than one phase function

- Consider two systems:
  System 1: pole at .75, zero at .5  System 2: pole at .75, zero at 2

- Comment: System 2 can be obtained by cascading System 1 with an all-pass system with a pole at .5 and a zero at 2. Hence the two systems have the same magnitude.
But what about the two phase responses?

Response of System 1:   \[ |H(\omega)|, \text{dB} \]

\[ \begin{array}{c}
\text{Comment: Systems have same magnitude, but System 2 has much greater phase shift}
\end{array} \]

Response of System 2:   \[ |H(\omega)|, \text{dB} \]

General comments on phase responses

- System 1 has much less phase shift than System 2; this is generally considered to be good
- System 1 has its zero inside unit circle; System 2 has zero its zero outside the unit circle
- A system is considered to be of minimum phase if all of its zeros lie inside the unit circle
- A system is considered to be of maximum phase if all of its zeros lie outside the unit circle
- Systems with more than one zero might have neither minimum nor maximum phase
A digression: Symmetry properties of DTFTs

Recall from DTFT properties:

If \( x[n] \Leftrightarrow X(e^{j\omega}) \)
then
\( x[-n] \Leftrightarrow X(e^{-j\omega}) \)
\( x^*[n] \Leftrightarrow X^*(e^{-j\omega}) \)
and ...
\( x[n] = x[-n] \Rightarrow X(e^{j\omega}) = X(e^{-j\omega}) \)
\( x[n] = -x[-n] \Rightarrow X(e^{j\omega}) = -X(e^{-j\omega}) \)
\( x[n] = x^*[n] \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega}) \)
real       Hermitian symmetric

Consequences of Hermitian symmetry

If \( X(e^{j\omega}) = X^*(e^{-j\omega}) \)
then
\( \text{Re}[X(e^{j\omega})] \) is even
\( \text{Im}[X(e^{j\omega})] \) is odd
\( |X(e^{j\omega})| \) is even
\( \angle X(e^{j\omega}) \) is odd

And
If \( x[n] \) is real and even, \( X(e^{j\omega}) \) will be real and even
and if \( x[n] \) is real and odd, \( X(e^{j\omega}) \) will be imaginary and odd
Zero phase systems

Consider an LSI system with an even unit sample response:

\[ H(z) = e^{2j\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-2j\omega} \]
\[ = 2\cos(2\omega) + 4\cos(\omega) + 3 \]

Comments:
- Frequency response is real, so system has zero phase shift
- This is to be expected since unit sample response is real and even

Linear phase systems

Now delay the system's sample response to make it causal:

\[ H(z) = e^{2j\omega} + 2e^{j\omega} + 3 + 3e^{-j\omega} + e^{-2j\omega} \]
\[ = e^{-2j\omega}(e^{2j\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-2j\omega}) \]
\[ = e^{-2j\omega}(2\cos(2\omega) + 4\cos(\omega) + 3) \]

Comment:
- Frequency response now exhibits linear phase shift
An additional comment or two

- The system on the previous page exhibits linear phase shift.
- This is also reasonable, since the corresponding sample response can be thought of as a zero-phase sample response that undergoes a time shift by two samples (producing a linear phase shift in the frequency domain).
- Another way to think about this is as a sample response that is even symmetric about the sample \( n = 2 \).
- Linear phase is generally considered to be more desirable than non-linear phase shift.
- If a linear-phase system is causal, it must be finite in duration. (The current example has only 5 nonzero samples.)

Another example of a linear phase systems

- Now let's consider a similar system but with an even number of sample points:

\[
H(z) = e^{j\omega} + 2e^{-j\omega} + 3e^{-2j\omega} + 3e^{-3j\omega} + 3e^{-4j\omega} + e^{-5j\omega}
= e^{-2.5j\omega}(e^{2.5j\omega} + 2e^{1.5j\omega} + 3e^{-5j\omega} + 3e^{-5j\omega} + 2e^{-1.5j\omega} + e^{-2.5j\omega})
= e^{-2.5j\omega}(2\cos(2.5\omega) + 4\cos(1.5\omega) + 3\cos(0.5\omega))
\]
Comments on the last system

- The system on the previous page also exhibits linear phase shift.
- In this case the corresponding sample response can be thought of as a zero-phase sample response that undergoes a time shift by 2.5 samples.
- In this case the unit sample response is symmetric about the point $n=2.5$.
- This type of system exhibits generalized linear phase, because the unit sample response is symmetric about a location that is between two integers.

Four types of linear-phase systems

- Oppenheim and Schafer refer to four types systems with generalized linear phase. All have sample points that are symmetric about its midpoint.
  - Type I: Odd number of samples, even symmetry
  - Type II: Even number of samples, even symmetry
  - Type III: Odd number of samples, odd symmetry
  - Type IV: Even number of samples, odd symmetry
Summary of second half of lecture

- All-pass systems have poles and zeros in mirror-image pairs
- Minimum phase causal and stable systems have all zeros (as well as all poles) inside the unit circle
- Maximum phase causal and stable systems have all zeros outside the unit circle
- Linear phase systems have unit sample responses that are symmetric about their midpoint (which may lie between two sample points)