18-660: Numerical Methods for Engineering Design and Optimization

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Electrical & Computer ENGINEERING

Overview

- Stochastic Optimization
  - Simulated annealing

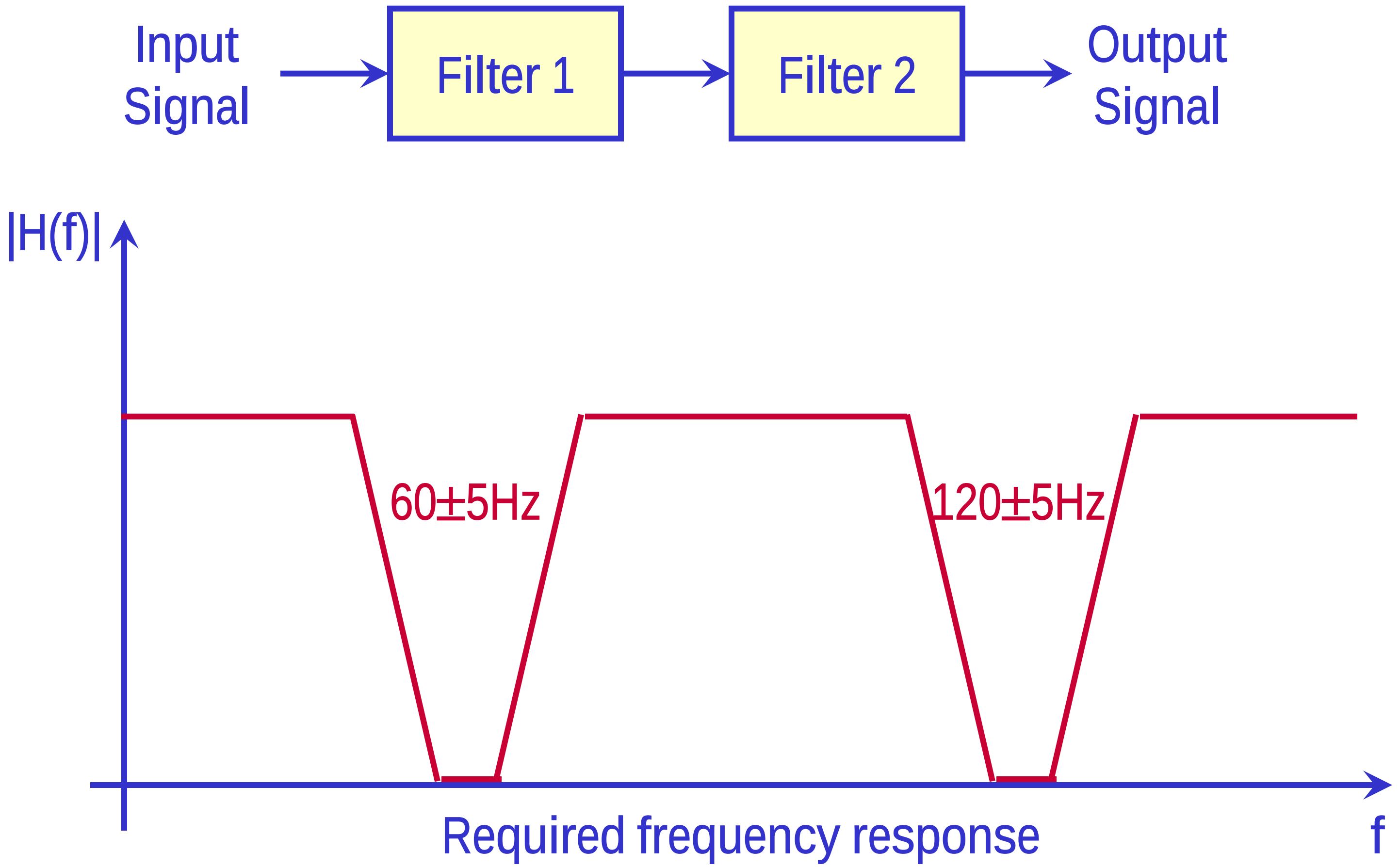
Local Optimization

- All optimization algorithms in early lectures assumes “local convexity” for cost function and constraint set
  - Gradient method
  - Newton method
  - Conjugate gradient method
  - Interior point method

- Global convergence cannot be guaranteed if the actual cost function or constraint set is non-convex

Filter Design Example

- Design a band-stop filter to remove power supply noise



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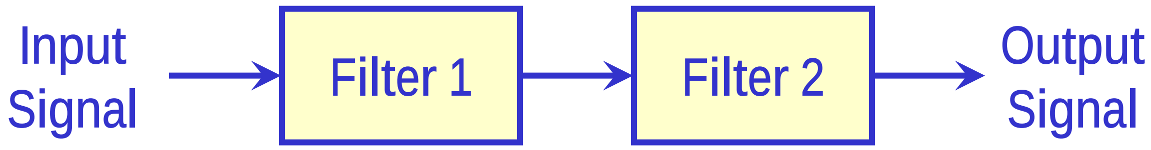

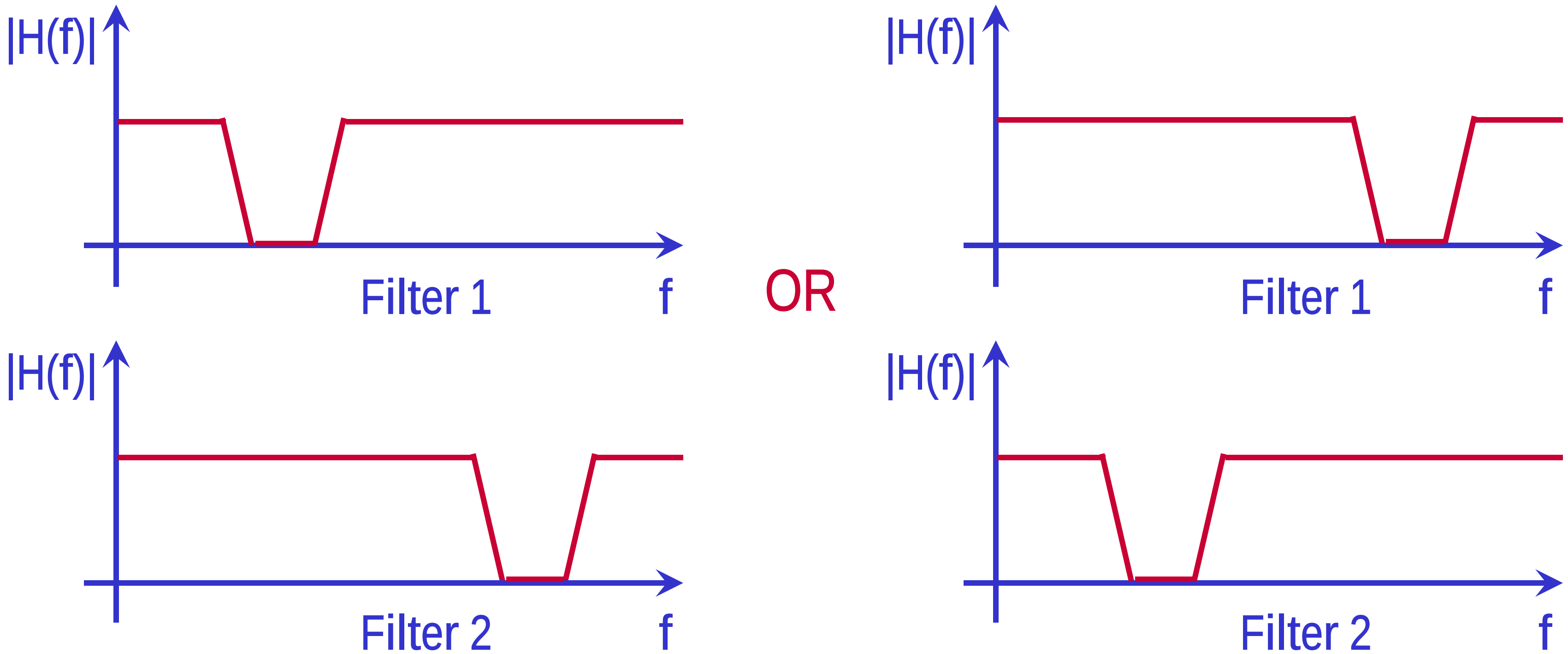

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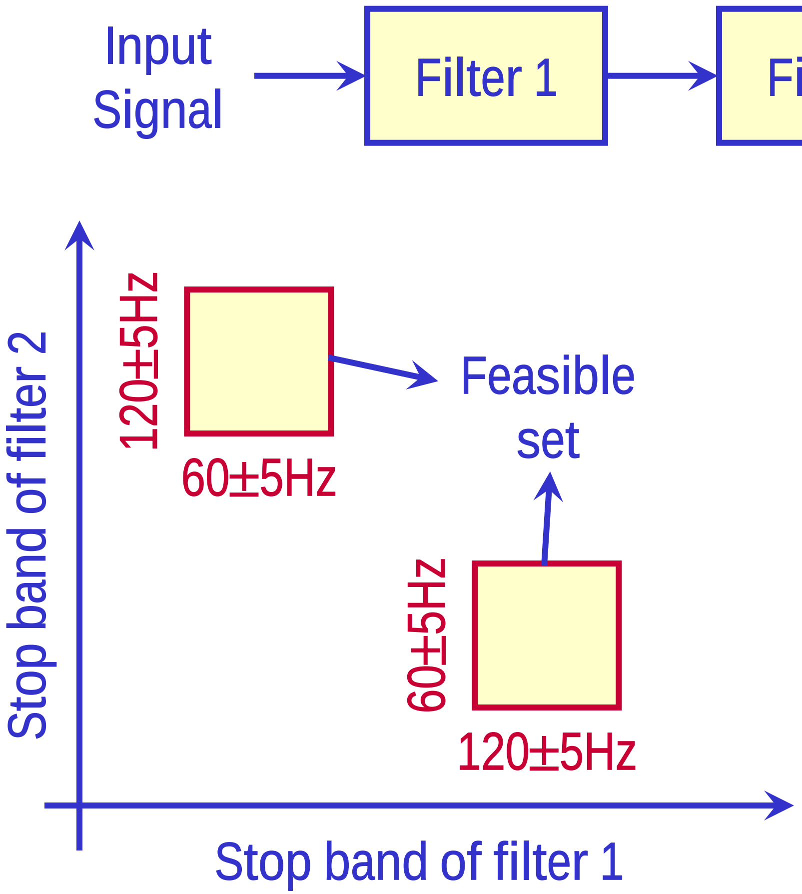


Feasible set is not continuous in this example!

Stochastic Optimization

- Stochastic optimization is another useful technique for nonlinear programming
  - Randomized algorithm (not deterministic)
  - Better convergence than local optimization
  - More expensive in computational cost

- Several important algorithms for stochastic optimization
  - Simulated annealing (focus of this lecture)
  - Genetic programming

Simulated Annealing

- Unconstrained optimization

$$\min_{X} \quad f(X)$$

- Simulated annealing:
  - Start from an initial point
  - Repeatedly consider various new solution points
  - Accept or reject some of these solution candidates
  - Converge to the optimal solution

Simulated Annealing

- Unconstrained optimization

$$\min_{X} \quad f(X)$$

- Simulated annealing was introduced by Metropolis in 1953

- It is based on “similarities” and “analogies” with the way that alloys manage to find a nearly global minimum energy level when they are cooled slowly

Simulated Annealing

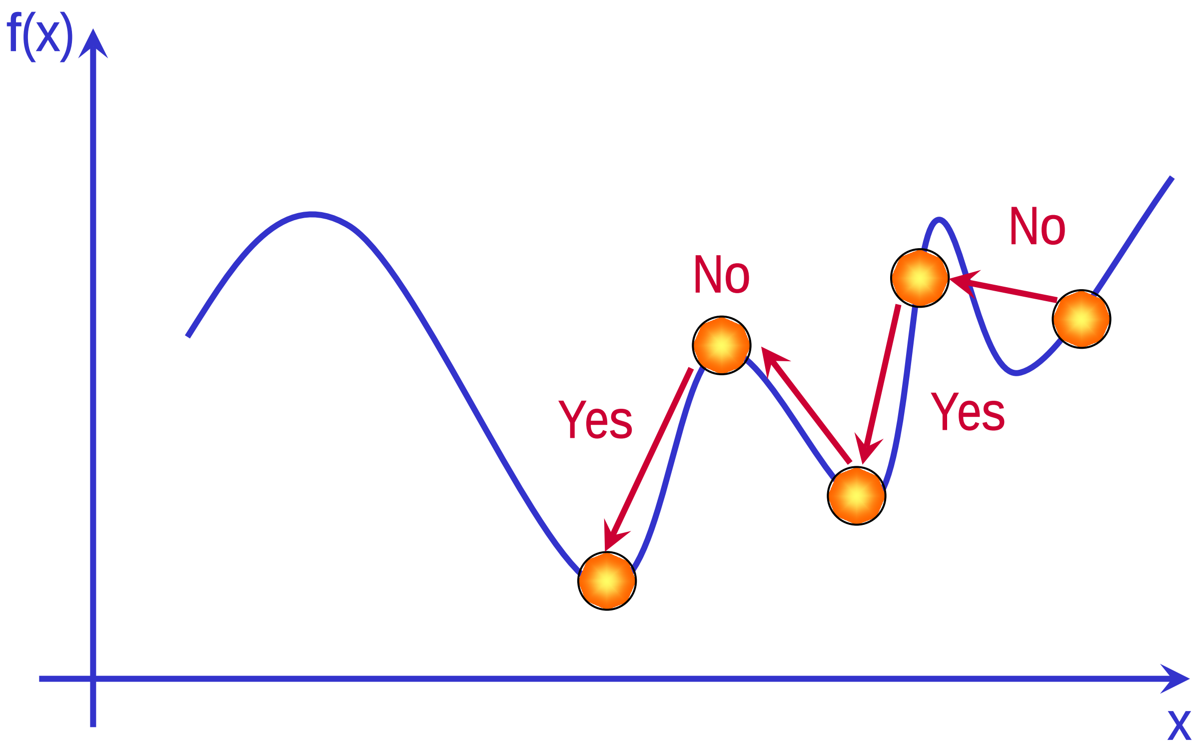
- Local optimization vs. simulated annealing

- Local optimization
  - Start from an initial point
  - Repeatedly consider various new solution points
  - Reduce cost function at each iteration
  - Converge to optimal solution

- Simulated annealing
  - Start from an initial point
  - Repeatedly consider various new solution points
  - Accept/reject new solution using probability at each iteration
  - Converge to optimal solution

Simulated Annealing

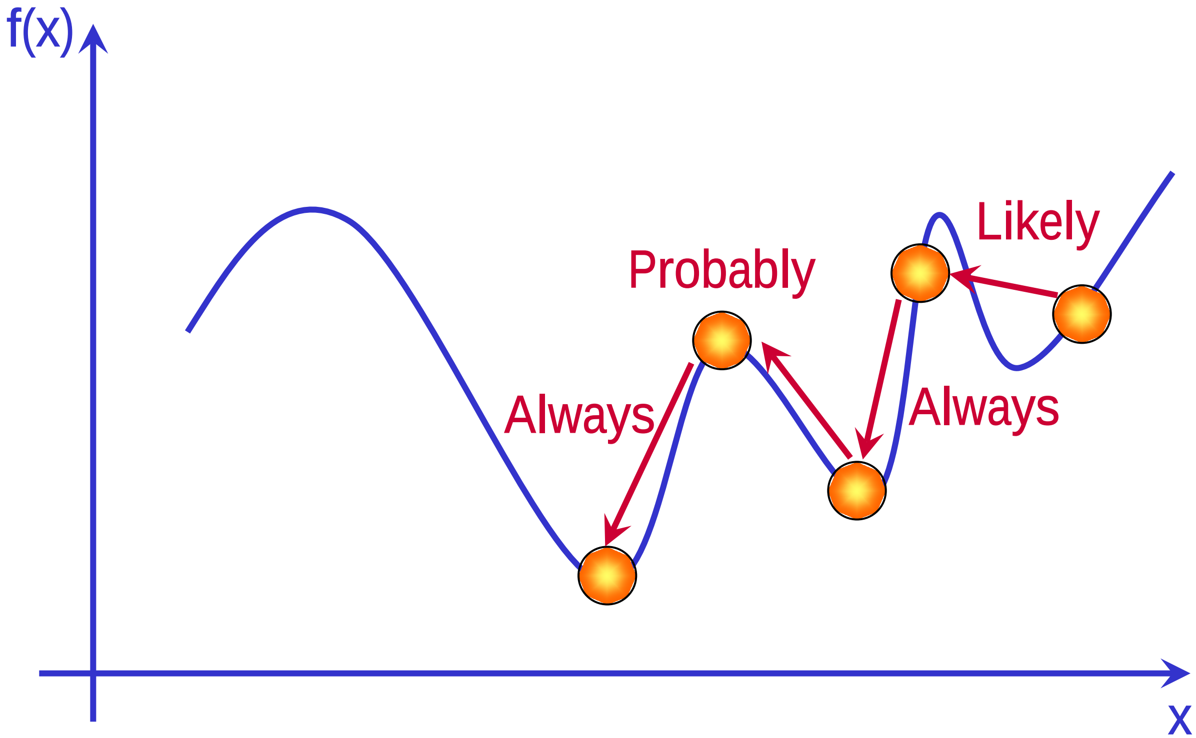
- Local optimization



Local optimization attempts to reduce cost function at each iteration

Simulated Annealing

- Simulated annealing



Simulated annealing accept/reject new solution candidate based on probability

Simulated Annealing

- Step 1: start from an initial point $X = X_0$ & $K = 0$
- Step 2: evaluate cost function $F = f(X_K)$
- Step 3: randomly move from X_K to a new solution X_{K+1}
- Step 4: if $f(X_{K+1}) < F$, then
  - Accept new solution
  - $X = X_{K+1}$ & $F = f(X_{K+1})$
- End if

} Similar to local optimization

- Step 5: if $f(X_{K+1}) \geq F$, then
  - Accept new solution with certain probability
  - $X = X_{K+1}$ & $F = f(X_{K+1})$ iff $rand(1) < \varepsilon$
- End if

} Help to get out of local minimum

- Step 6: $K = K + 1$ & go to Step 2

Simulated Annealing

- Accept/reject new solution with the probability **ε**
  - If $f(X_{K+1}) \geq F$, then
    - Accept new solution with certain probability
    - $X = X_{K+1}$ & $F = f(X_{K+1})$ iff rand(1) $< \varepsilon$
  - End if

- Option 1
  - Constant probability, i.e., $\varepsilon = 0.1$

- Option 2 (better than Option 1)
  - Dynamically varying probability, i.e., decreasing over time

Simulated Annealing

- Accept/reject new solution with the probability ε
  - If $f(X_{K+1}) \geq F$, then
    - Accept new solution with certain probability
    - $X = X_{K+1}$ & $F = f(X_{K+1})$ iff $rand(1) < \varepsilon$
  - End if

- Use Boltzmann distribution to determine the probability ε

$$\varepsilon = \exp\left[-\frac{f\left(X_{K+1}\right) - F}{T_{K+1}} \right]$$

  - T_{K+1} is a "temperature" parameter that gradually decreases
  - E.g., $T_{K+1} = \alpha \cdot T_K$ where $\alpha < 1$

Simulated Annealing

- Accept/reject new solution with the probability **ε**
  - If $f(X_{K+1}) \geq F$, then
    - Accept new solution with certain probability
    - $X = X_{K+1}$ & $F = f(X_{K+1})$ iff

$$\mathrm{rand}(1) \leq \exp\left[-\frac{f(X_{K+1}) - F}{T_{K+1}}\right]$$

  - End if

- High temperature
  - Attempt to accept all new solutions even if $f(X_{K+1}) - F$ is large

- Low temperature
  - Only accept the new solutions where $f(X_{K+1}) - F$ is small

Simulated Annealing

- Simulated annealing is particularly developed for unconstrained optimization

- Constrained optimization can be converted to unconstrained optimization using barrier method

$$\begin{aligned} \min_X \quad & f(X) \\ \text{S.T.} \quad & g(X) \le 0 \end{aligned} \quad \Rightarrow \quad \min_X \quad f(X) - \frac{1}{t} \cdot \log[-g(X)]$$

Simulated Annealing

- Simulated annealing does not guarantee global optimum
  - However, it tries to avoid a large number of local minima
  - Therefore, it often yields a better solution than local optimization
- Simulated annealing is not deterministic
  - Whether accept or reject a new solution is random
  - You can get different answers from multiple runs
- Simulated annealing is more expensive than local optimization
  - It is the price you must pay to achieve a better optimal solution

Simulated Annealing

- Simulated annealing has been used to solve many practical engineering problems

- A large number of implementation issues must be considered for practical circuit optimization problems
  - How to define optimization variable X (continuous vs. discrete)?
  - How to randomly move to a new solution?
  - Etc.

Example: Travelling Salesman Problem (TSP)

- N cities are located on a 2-D map
- One must visit each city once and then return to start city
- Find the optimal route with minimum length
  - If all cities are visited in the order of R = {C_1, C_2, ..., C_N}, we have

$$f(R) = \|C_1 - C_2\|_2 + \|C_2 - C_3\|_2 + \cdots + \|C_N - C_1\|_2$$

Distance between C_1 and C_2

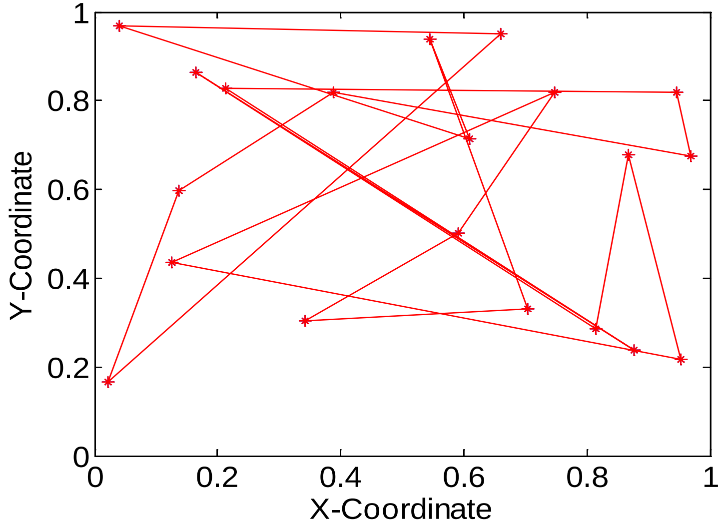
Distance between C_N and C_1

Example: Travelling Salesman Problem (TSP)

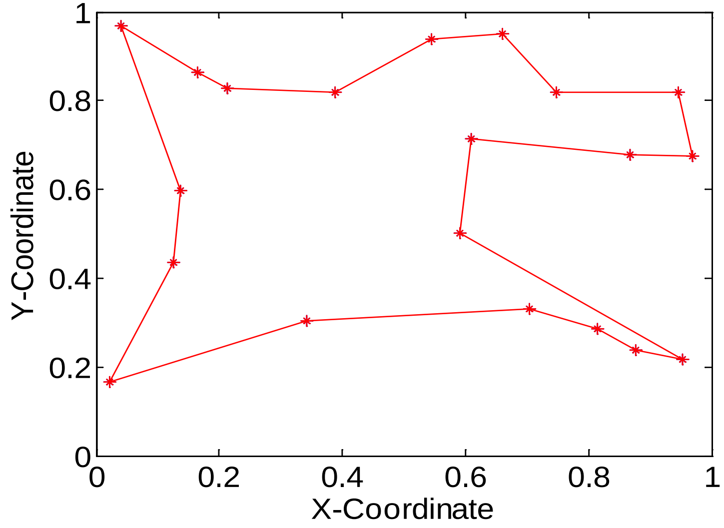
- Step 1: start from random route R, initial temperature T & K = 1
- Step 2: evaluate cost function F = f(R)
- Step 3: define new route R_K by randomly swapping two cities
- Step 4: if $f(R_K) < F$, then
  - Accept new route
  - $R = R_K$ & $F = f(R_K)$
- End if
- Step 5: if $f(R_K) \geq F$, then
  - Accept new solution with certain probability
  - $R = R_K$ & $F = f(R_K)$ iff $\mathrm{rand}(1) < \exp\{[F - f(R_K)]/T\}$
- End if
- Step 6: $T = \alpha T$ ($\alpha < 1$), $K = K + 1$, and go to Step 3

Example: Travelling Salesman Problem (TSP)

- TSP route optimized by simulated annealing



Initial route



Optimized route

Summary

- Stochastic optimization
  - Simulated annealing