18-660: Numerical Methods for Engineering Design and Optimization

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Electrical & Computer ENGINEERING

Overview

- Constrained Optimization
  - Inequality constraint
  - Interior point method
  - Feasibility problem

Inequality Constrained Optimization

$$\min_{X} \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1, 2, \cdots, M)$$
$$AX = B$$

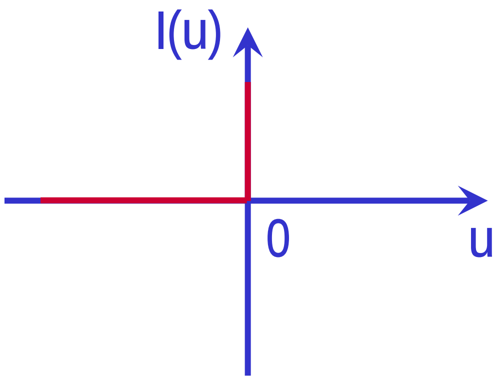
- Equality constraint can be written as two inequality constraints

$$g(X) = 0 \quad \Rightarrow \quad \begin{aligned} g(X) &\le 0 \\ -g(X) &\le 0 \end{aligned}$$

Indicator Function

- Define indicator function

$$I(u)=\begin{cases}0 & (u\le 0)\\ +\infty & (u>0)\end{cases}$$



$$\begin{aligned}&\min_X \quad f(X)\\ &\text{S.T.} \quad g_m(X)\le 0 \quad (m=1,2,\cdots,M)\\ &\qquad\quad AX=B\end{aligned}$$

$$\begin{aligned}&\min_X \quad f(X)+\sum_{m=1}^{M} I[g_m(X)]\\ &\text{S.T.} \quad AX=B\end{aligned}$$

$g_m(X)$ must be less than 0 so that the cost function does not reach inf

Indicator Function

$$\min_{X} \quad f(X) + \sum_{m=1}^{M} I[g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

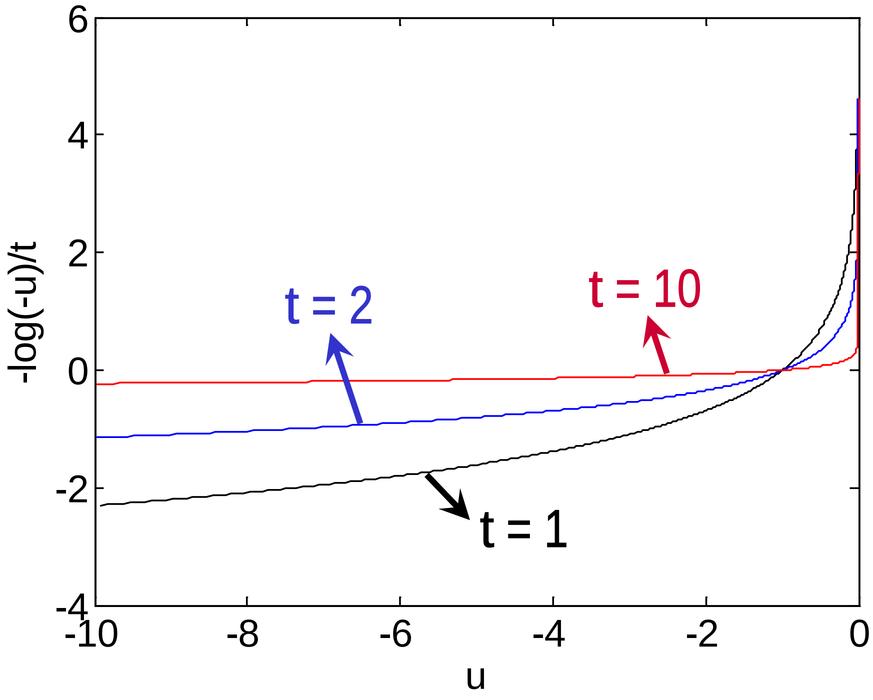
- Result in a new optimization problem with linear constraints only
  - However, the indicator function I(•) is not smooth
  - We cannot directly apply Lagrange multiplier and calculate 1st/2nd-order derivatives
- New idea: approximate I(•) by a smooth function

Logarithmic Barrier

- Approximate I(•) by logarithmic barrier

$$I(u) \approx -1/t \cdot \log(-u) \quad (u \le 0)$$

  - where t > 0 is a user-defined parameter



Logarithmic barrier converges to I(•) iff $t \rightarrow \infty$

Logarithmic Barrier

$$\min_X \quad f(X) + \sum_{m=1}^{M} I[g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

$$I(u) \approx -1/t \cdot \log(-u) \quad (u \le 0)$$

$$\min_X \quad f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

- Open question: does the new optimization preserve convexity?

Logarithmic Barrier

$$\begin{aligned} \min_X \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1, 2, \cdots, M) \\ & AX = B \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_X \quad & f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)] \\ \text{S.T.} \quad & AX = B \end{aligned}$$

- If f(X) and g_m(X) are convex

$$\varphi_m(X) = -\frac{1}{t} \cdot \log[-g_m(X)]$$

$$\nabla \varphi_m(X) = -\frac{1}{t} \cdot \frac{1}{-g_m(X)} \cdot [-\nabla g_m(X)] = -\frac{1}{t} \cdot \frac{1}{g_m(X)} \cdot \nabla g_m(X)$$

Logarithmic Barrier

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M)$$
$$AX = B$$

$$\Rightarrow$$

$$\min_X \quad f(X) - \frac{1}{t}\cdot\sum_{m=1}^{M}\log[-g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

- If f(X) and g_m(X) are convex

$$\nabla\varphi_m(X) = -\frac{1}{t}\cdot\frac{1}{g_m(X)}\cdot\nabla g_m(X)$$

$$\underbrace{\nabla^2\varphi_m(X)}_{\text{Positive semi-definite}} = -\frac{1}{t}\cdot\frac{1}{-[g_m(X)]^2}\cdot\nabla g_m(X)\nabla g_m(X)^T - \frac{1}{t}\cdot\frac{1}{g_m(X)}\cdot\nabla^2 g_m(X)$$

$$= \frac{1}{t}\cdot\frac{1}{[g_m(X)]^2}\cdot\underbrace{\nabla g_m(X)\nabla g_m(X)^T}_{\text{Positive semi-definite}} - \frac{1}{t}\cdot\underbrace{\frac{1}{g_m(X)}}_{\text{Negative}}\cdot\underbrace{\nabla^2 g_m(X)}_{\text{Positive semi-definite}}$$

Logarithmic Barrier

$$\begin{aligned} \min_X \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1, 2, \cdots, M) \\ & AX = B \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_X \quad & f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)] \\ \text{S.T.} \quad & AX = B \end{aligned}$$

- If f(X) and g_m(X) are convex

$$\varphi_m(X) = -\frac{1}{t} \cdot \log[-g_m(X)]$$

$$\begin{aligned} \min_X \quad & f(X) + \sum_{m=1}^{M} \varphi_m(X) \longrightarrow \text{Convex cost function} \\ \text{S.T.} \quad & AX = B \end{aligned}$$

Interior Point Method

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1, 2, \cdots, M) \\ & AX = B \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{X} \quad & f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)] \\ \text{S.T.} \quad & AX = B \end{aligned}$$

- Interior point method is also referred to as barrier method
  - Step 1: select an initial value of t and an initial guess $X^{(0)}$
  - Step 2: solve linear equality constrained nonlinear optimization to find the optimal solution X*
  - Step 3: $X^{(0)}$ = X* and t = βt (β is typically 10~20)
  - Repeat Step 2~3 until t is sufficiently large

Feasibility Problem

$$\min_X \quad f(X) \qquad \text{S.T.} \quad g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \qquad AX = B$$

$$\Rightarrow \quad \min_X \quad f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)] \qquad \text{S.T.} \quad AX = B$$

- When we iteratively solve linear equality constrained nonlinear optimization, $X^{(0)}$ must be feasible

$$AX^{(0)} = B \quad g_m\left[X^{(0)}\right] \le 0 \quad (m = 1,2,\cdots,M)$$

  - Otherwise, $\log\{-g_m[X^{(0)}]\}$ does not have a numerical value
  - We cannot move to the next iteration step

Feasibility Problem

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1, 2, \cdots, M)$$
$$AX = B$$

$$\Rightarrow$$

$$\min_X \quad f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

- How do we come up with an initial feasible solution?

$$\underbrace{AX^{(0)} = B}_{\text{Easy}} \quad \underbrace{g_m\left[X^{(0)}\right] \le 0 \quad (m = 1, 2, \cdots, M)}_{\text{Difficult}}$$

- How do we even know that the optimization is feasible?
  - Not all optimization problems have a feasible solution

Phase I Method

$$\min_X \quad f(X)$$
$$\text{S.T.} \quad g_m(X) \le 0 \quad (m = 1, 2, \cdots, M)$$
$$AX = B$$

$$\Rightarrow$$

$$\min_X \quad f(X) - \frac{1}{t} \cdot \sum_{m=1}^{M} \log[-g_m(X)]$$
$$\text{S.T.} \quad AX = B$$

- We must do another optimization to decide
  - Is the optimization feasible?
  - If yes, find one of the feasible solutions

- This preprocessing step is called phase I, and the interior point method should be applied for phase II

Phase I Method

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1,2,\cdots,M) \\ & AX = B \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{X,s} \quad & s \\ \text{S.T.} \quad & g_m(X) \le s \quad (m = 1,2,\cdots,M) \\ & AX = B \end{aligned}$$

Phase II problem — Phase I problem

- Once optimal point [X* s*] is found for phase I problem, we know:
  - If s* > 0
    - Phase II problem is not feasible
  - If s* ≤ 0
    - Phase II problem is feasible
    - X* is one of the feasible solutions
    - Starting from X*, apply interior point method to solve phase II problem

Phase I Method

$$\begin{aligned} \min_{X} \quad & f(X) \\ \text{S.T.} \quad & g_m(X) \le 0 \quad (m = 1, 2, \cdots, M) \\ & AX = B \end{aligned}$$

Phase II problem

$$\begin{aligned} \min_{X,s} \quad & s \\ \text{S.T.} \quad & g_m(X) \le s \quad (m = 1, 2, \cdots, M) \\ & AX = B \end{aligned}$$

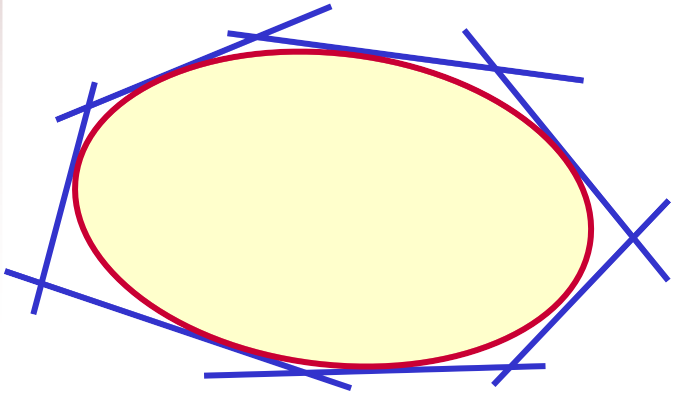
Phase I problem

- Phase I problem can be easily solved
  - Select an initial $X^{(0)}$ that satisfies $AX^{(0)} = B$
  - Calculate $g_m[X^{(0)}]$ where m = 1,2,...,M
  - Determine the maximum value of $g_m[X^{(0)}]$, denoted as g_{MAX}
  - Set $s^{(0)} = g_{MAX}$
  - Starting from $[X^{(0)}; s^{(0)}]$, apply interior point method to solve phase I problem and find its optimal solution $[X^*; s^*]$

Semidefinite Programming

- Inequality constraints are not always represented as g(X) ≤ 0

- Example: maximum inscribed ellipsoid
  - Generalized inequality can be solved by semidefinite programming



$$\max_{W,d} \quad \det(W)$$

$$S.T. \quad \|WB_k\|_2 + B_k^T \cdot d + C_k \leq 0 \quad (k = 1, 2, \cdots, K)$$

$$W = W^T$$

$$W \succ= 0$$

Generalized inequality

Semidefinite Programming

$$\min_{X} \quad f(X)$$

$$g(X) = \underline{x_1 F_1 + x_2 F_2 + \cdots + x_M F_M} \succ= 0$$

- Define logarithmic barrier function

$$\varphi(X) = -\frac{1}{t} \cdot \log\left[\det\left(x_1 F_1 + x_2 F_2 + \cdots + x_M F_M\right)\right]$$

- φ(X) is convex
  - log[det(•)] is concave
  - –log[det(•)] is convex

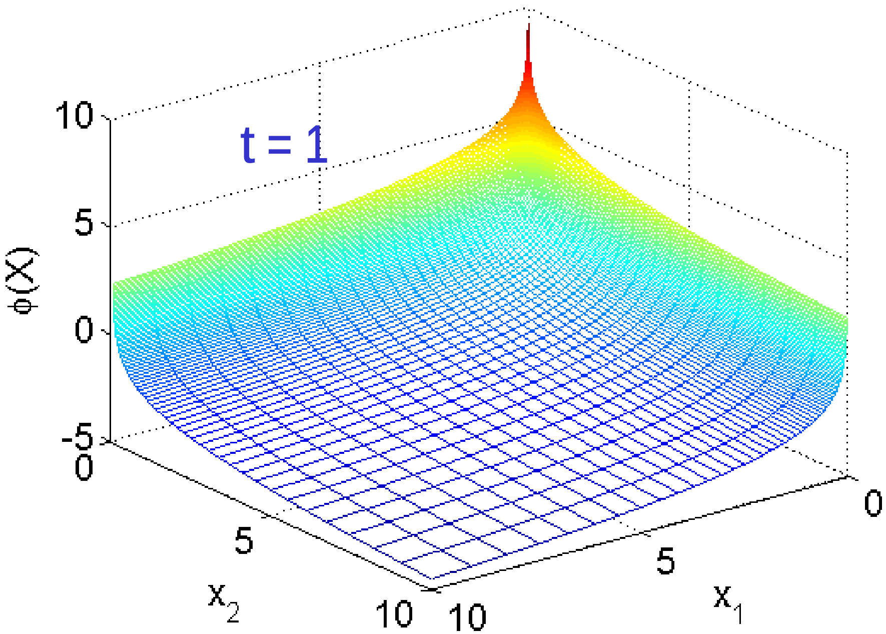
Semidefinite Programming

$$\varphi(X) = -\frac{1}{t}\cdot\log\left[\det\left(\underbrace{x_1F_1 + x_2F_2 + \cdots + x_MF_M}_{g(X)}\right)\right]$$

- φ(X) approaches infinite, if g(X) becomes indefinite

$$g(X) = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

$x_1 \geq 0$ and $x_2 \geq 0$ so that g(X) is positive semidefinite



Semidefinite Programming

$$\min_{X} \quad f(X)$$
$$x_1 F_1 + \cdots + x_M F_M \succ= 0$$

Phase II problem

$$\Rightarrow$$

$$\min_{X,s} \quad s$$
$$x_1 F_1 + \cdots + x_M F_M + sI \succ= 0$$

Phase I problem

- Phase I method
  - Arbitrarily select an initial $X^{(0)}$
  - Select a sufficiently large $s^{(0)}$ so that phase I constraint is feasible
  - Starting from [$X^{(0)}$; $s^{(0)}$], apply interior point method to solve phase I problem and find its optimal solution [X*; s*]

What value of $s^{(0)}$ is sufficiently large?

Semidefinite Programming

- A matrix F is diagonally dominant if

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots \\ F_{21} & F_{22} & F_{23} & \cdots \\ F_{31} & F_{32} & F_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{gathered} |F_{11}| \geq |F_{12}| + |F_{13}| + \cdots \\ |F_{22}| \geq |F_{21}| + |F_{23}| + \cdots \\ |F_{33}| \geq |F_{31}| + |F_{32}| + \cdots \\ \vdots \end{gathered}$$

- A matrix F is positive semidefinite, if
  - F is symmetric, and
  - F is diagonally dominant, and
  - All diagonal elements are non-negative

Semidefinite Programming

$$\underbrace{x_1^{(0)}F_1 + \cdots + x_M^{(0)}F_M}_{g(X)} + s^{(0)}I \succ= 0$$

- Select a sufficiently large value of $s^{(0)}$ so that the matrix $g[X^{(0)}] + s^{(0)}I$ is diagonally dominant (hence, positive semidefinite)

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots \\ g_{21} & g_{22} & g_{23} & \cdots \\ g_{31} & g_{32} & g_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} s^{(0)} & & & \\ & s^{(0)} & & \\ & & s^{(0)} & \\ & & & \ddots \end{bmatrix}$$

Summary

- Constrained optimization
  - Inequality constraint
  - Interior point method
  - Feasibility problem