18-660: Numerical Methods for Engineering Design and Optimization

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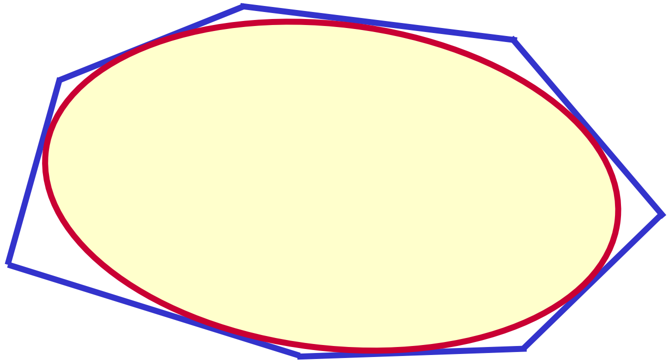
Electrical & Computer ENGINEERING

Overview

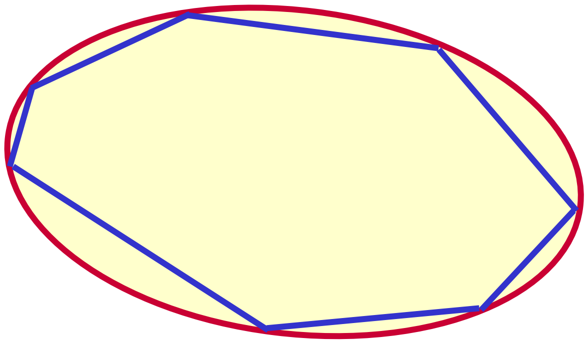
- Geometric Problems
  - Maximum inscribed ellipsoid
  - Minimum circumscribed ellipsoid

Geometric Problems

- Many geometric problems can be solved by convex programming
  - Consider maximum inscribed ellipsoid as an example
  - Derive mathematical formulation as convex optimization



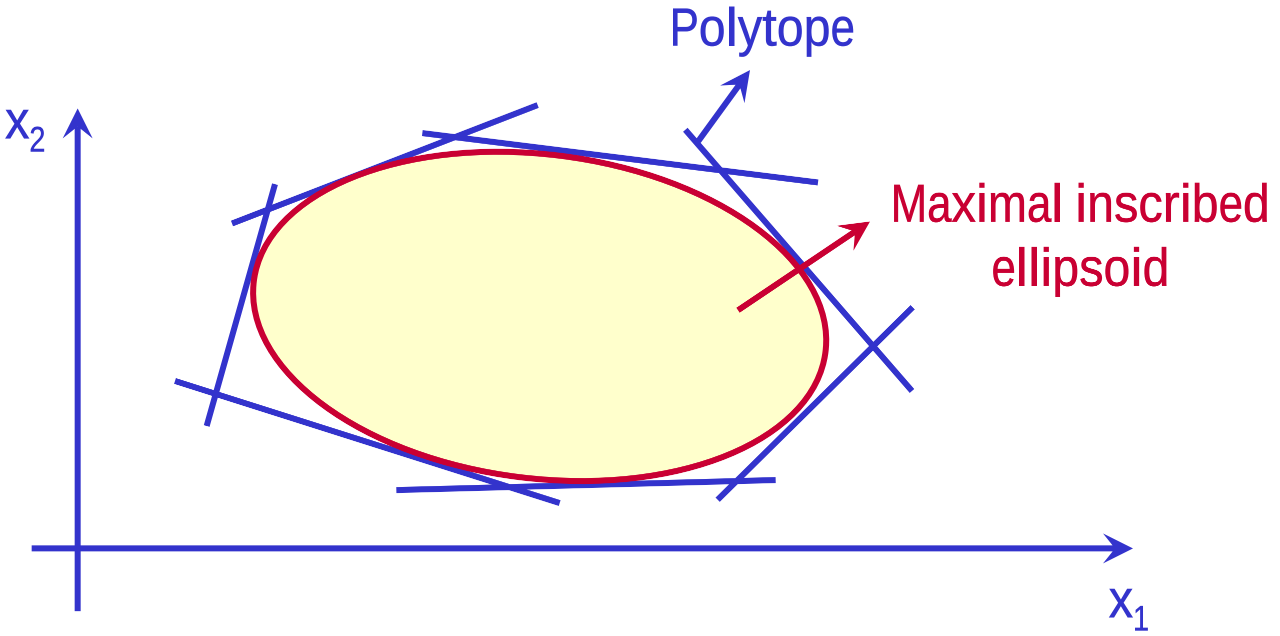
Maximum inscribed ellipsoid



Minimum circumscribed ellipsoid

Maximum Inscribed Ellipsoid

- Problem definition:
  - Given a bounded polytope, find the inscribed ellipsoid that has the maximal volume



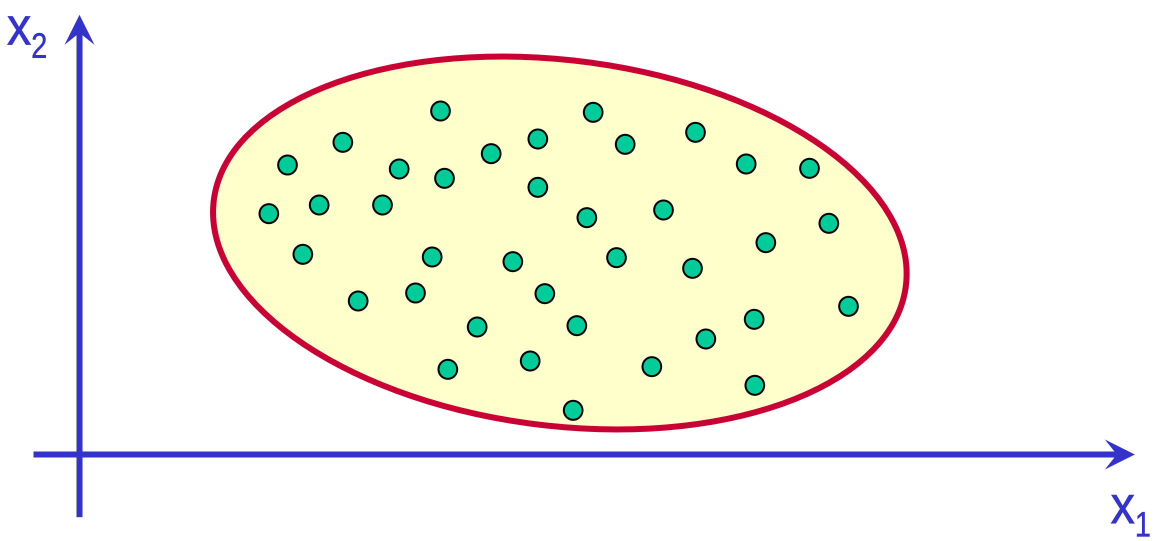
Maximum Inscribed Ellipsoid

- Representation of an ellipsoid

$$\Omega = \left\{ X = W \cdot u + d \mid \|u\|_2 \le 1 \right\}$$ A set of points inside the ellipsoid

  - W is symmetric and positive definite
  - $\|u\|_2$ denotes the L_2-norm of a vector

$$\|u\|_2 = \sqrt{u^T u}$$ Length of the vector u



Maximum Inscribed Ellipsoid

- Representation of an ellipsoid

$$\Omega = \left\{ X = W \cdot u + d \mid \|u\|_2 \le 1 \right\}$$

⇩

$$u = W^{-1} \cdot (X - d)$$

⇩

$$\|u\|_2^2 = u^T u = \left[W^{-1} \cdot (X - d) \right]^T \cdot \left[W^{-1} \cdot (X - d) \right] \le 1$$

⇩

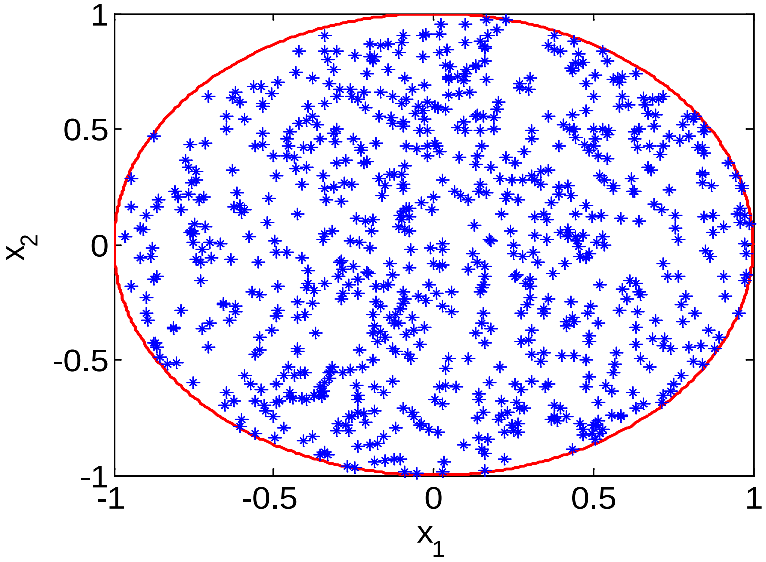
$$\underbrace{(X - d)^T \cdot W^{-T} \cdot W^{-1} \cdot (X - d)}_{\text{Quadratic function of X}} \le 1$$

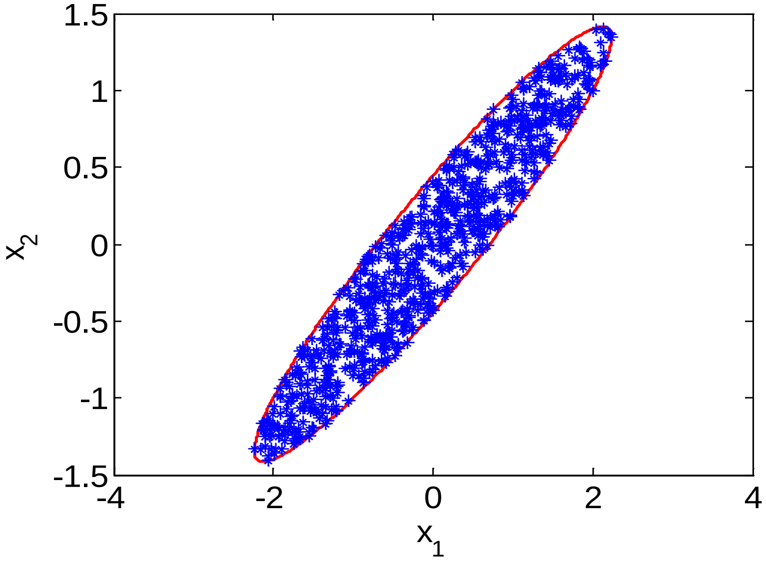
Quadratic function of X

Maximum Inscribed Ellipsoid

- Two examples

$$\Omega = \left\{ X = W \cdot u + d \mid \|u\|_2 \leq 1 \right\}$$





$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$d = 0$$

$$W = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$d = 0$$

Maximum Inscribed Ellipsoid

- Given the ellipsoid

$$\Omega = \left\{ X = W \cdot u + d \mid \|u\|_2 \leq 1 \right\}$$

  - and the polytope

$$B_k^T X + C_k \leq 0 \quad (k = 1, 2, \cdots, K)$$

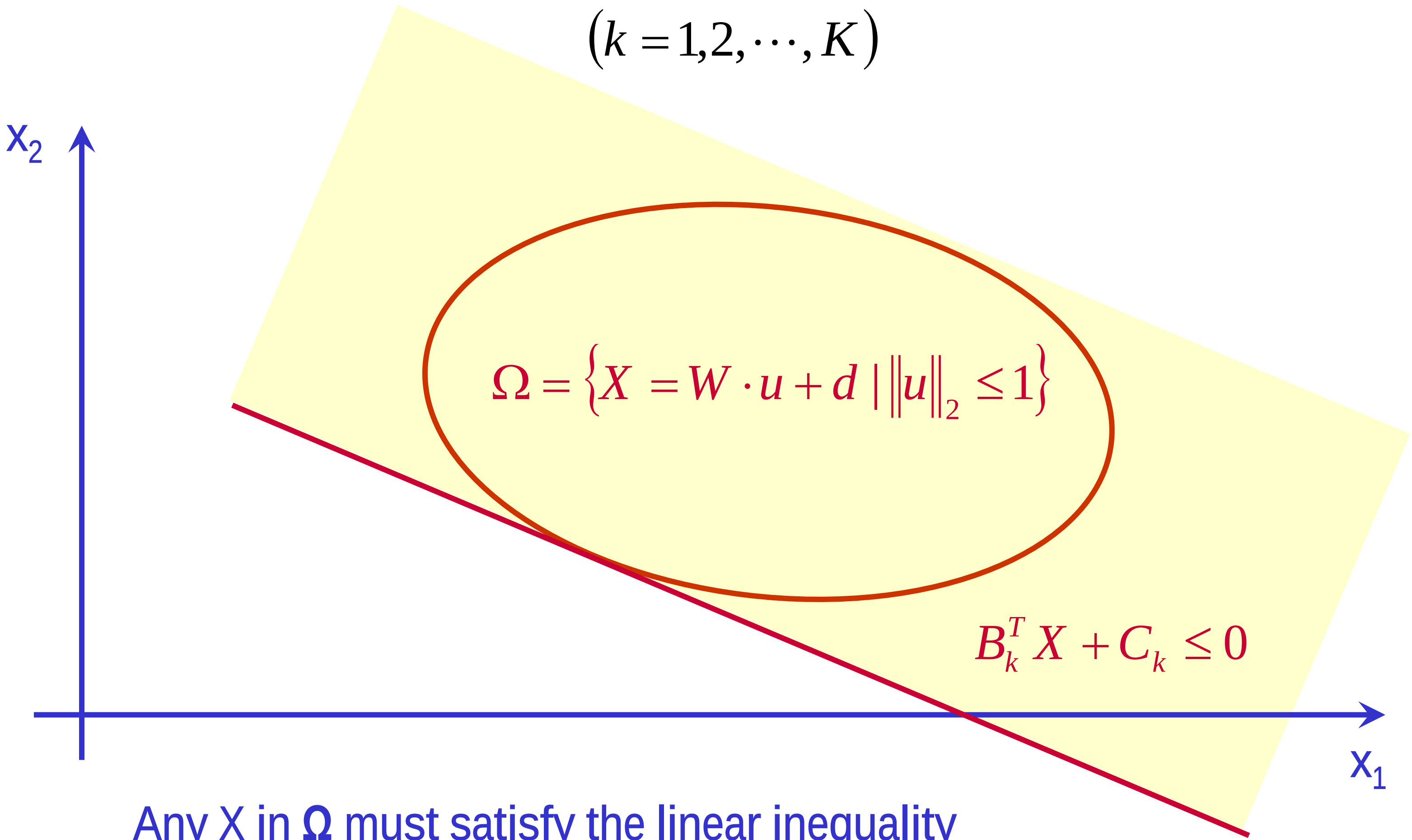
  - we require that any X in Ω satisfies the linear inequality

$$B_k^T X + C_k \leq 0 \quad (\forall X \in \Omega)$$
$$(k = 1, 2, \cdots, K)$$

Basic condition for inscribed ellipsoid

Maximum Inscribed Ellipsoid

$$B_k^T X + C_k \le 0 \quad (\forall X \in \Omega)$$
$$(k = 1, 2, \cdots, K)$$



Any X in **Ω** must satisfy the linear inequality

Maximum Inscribed Ellipsoid

$$\Omega = \left\{ X = W \cdot u + d \mid \|u\|_2 \leq 1 \right\}$$

$$B_k^T X + C_k \leq 0 \quad \left(\forall X \in \Omega \right)$$
$$\left(k = 1, 2, \cdots, K \right)$$

⇩

$$B_k^T \cdot W \cdot u + B_k^T \cdot d + C_k \leq 0 \quad \left(\forall \|u\|_2 \leq 1 \right)$$
$$\left(k = 1, 2, \cdots, K \right)$$

⇩

$$\sup_{\|u\|_2 \leq 1} \left(B_k^T \cdot W \cdot u + B_k^T \cdot d + C_k \right) \leq 0 \quad \left(k = 1, 2, \cdots, K \right)$$

sup(●) denotes the supremum (i.e., the least upper bound) of a set

Maximum Inscribed Ellipsoid

$$\sup_{\|u\|_2 \le 1} \left(B_k^T \cdot W \cdot u + B_k^T \cdot d + C_k \right) \le 0$$

⇓

$$\sup_{\|u\|_2 \le 1} \left(B_k^T \cdot W \cdot u \right) + B_k^T \cdot d + C_k \le 0$$

⇓ $p = W^T B_k$

$$\sup_{\|u\|_2 \le 1} \left(p^T \cdot u \right) + B_k^T \cdot d + C_k \le 0$$

Inner product of two vectors

Maximum Inscribed Ellipsoid

- Inner product of two vectors

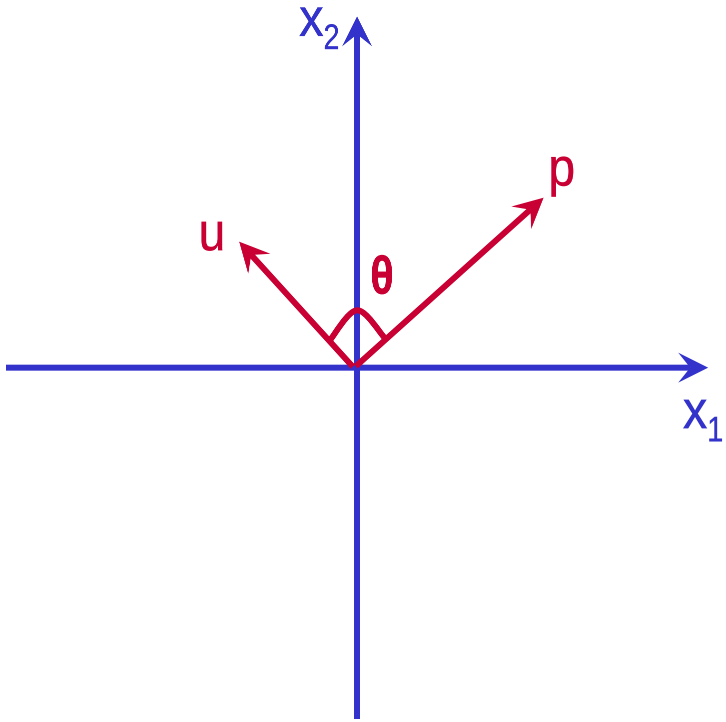
$$p = \begin{bmatrix} p_1 & p_2 & \cdots \end{bmatrix}^T$$
$$u = \begin{bmatrix} u_1 & u_2 & \cdots \end{bmatrix}^T$$

⇩

$$p^T \cdot u = p_1 u_1 + p_2 u_2 + \cdots$$
$$= \|p\|_2 \cdot \|u\|_2 \cdot \cos\theta$$

⇩

$$\sup_{\|u\|_2 \le 1} \left(p^T \cdot u\right) = \|p\|_2 \cdot 1 \cdot \cos 0 = \|p\|_2$$



Maximum Inscribed Ellipsoid

$$\sup_{\|u\|_2 \le 1} \left(p^T \cdot u\right) + B_k^T \cdot d + C_k \le 0 \qquad \sup_{\|u\|_2 \le 1} \left(p^T \cdot u\right) = \|p\|_2$$

⇓

$$\|p\|_2 + B_k^T \cdot d + C_k \le 0$$

⇓ $p = W^T B_k$

$$\left\|W^T B_k\right\|_2 + B_k^T \cdot d + C_k \le 0 \quad \left(k = 1, 2, \cdots, K\right)$$

Inequality constraints

Maximum Inscribed Ellipsoid

- Given these inequality constraints, we want to maximize the ellipsoid volume

$$\Omega = \left\{ X = W \cdot u + d \mid \|u\|_2 \leq 1 \right\}$$

⇩

$$Volume \propto \det(W)$$

Why is volume proportional to the determinant of W?

Maximum Inscribed Ellipsoid

- If W is diagonal, we have

$$X = W \cdot u + d \quad \left(\|u\|_2 \le 1 \right)$$

$$\Downarrow$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} \cdot u + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \left(\|u\|_2 \le 1 \right)$$

$$\Downarrow \quad (X - d)^T \cdot W^{-T} \cdot W^{-1} \cdot (X - d) \le 1$$

$$\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right)^T \cdot \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & W_{22}^{-1} \end{bmatrix} \cdot \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & W_{22}^{-1} \end{bmatrix} \cdot \left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right) \le 1$$

Maximum Inscribed Ellipsoid

- If W is diagonal, we have

$$\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\right)^T \cdot \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & W_{22}^{-1} \end{bmatrix} \cdot \begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & W_{22}^{-1} \end{bmatrix} \cdot \left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\right) \leq 1$$

$$\Downarrow$$

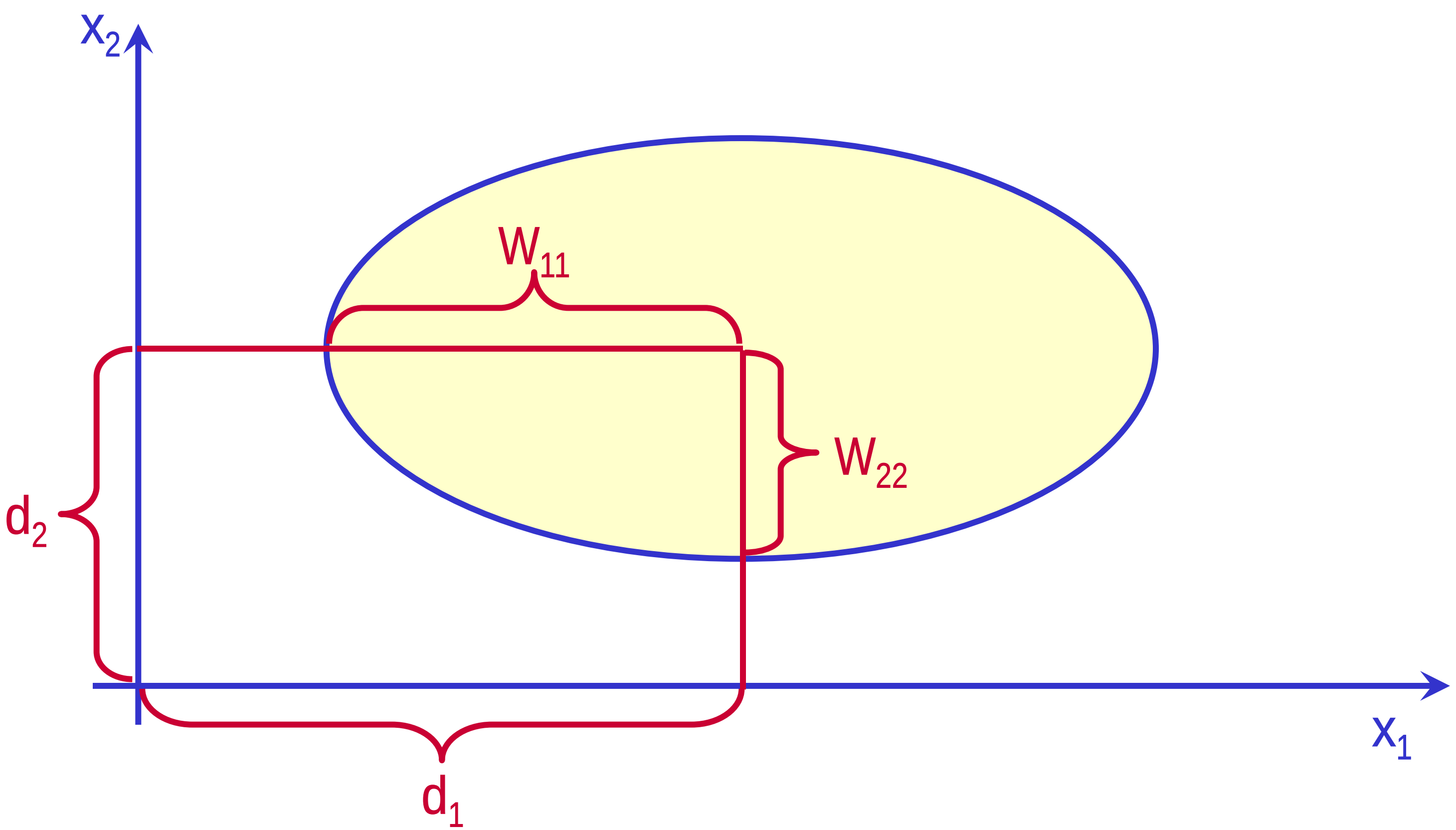
$$\frac{(X_1 - d_1)^2}{W_{11}^2} + \frac{(X_2 - d_2)^2}{W_{22}^2} \leq 1$$

  - d_1 and d_2 determine the center of the ellipsoid
  - W_{11} and W_{22} determine the semi-axes of the ellipsoid

Maximum Inscribed Ellipsoid

- If W is diagonal, we have

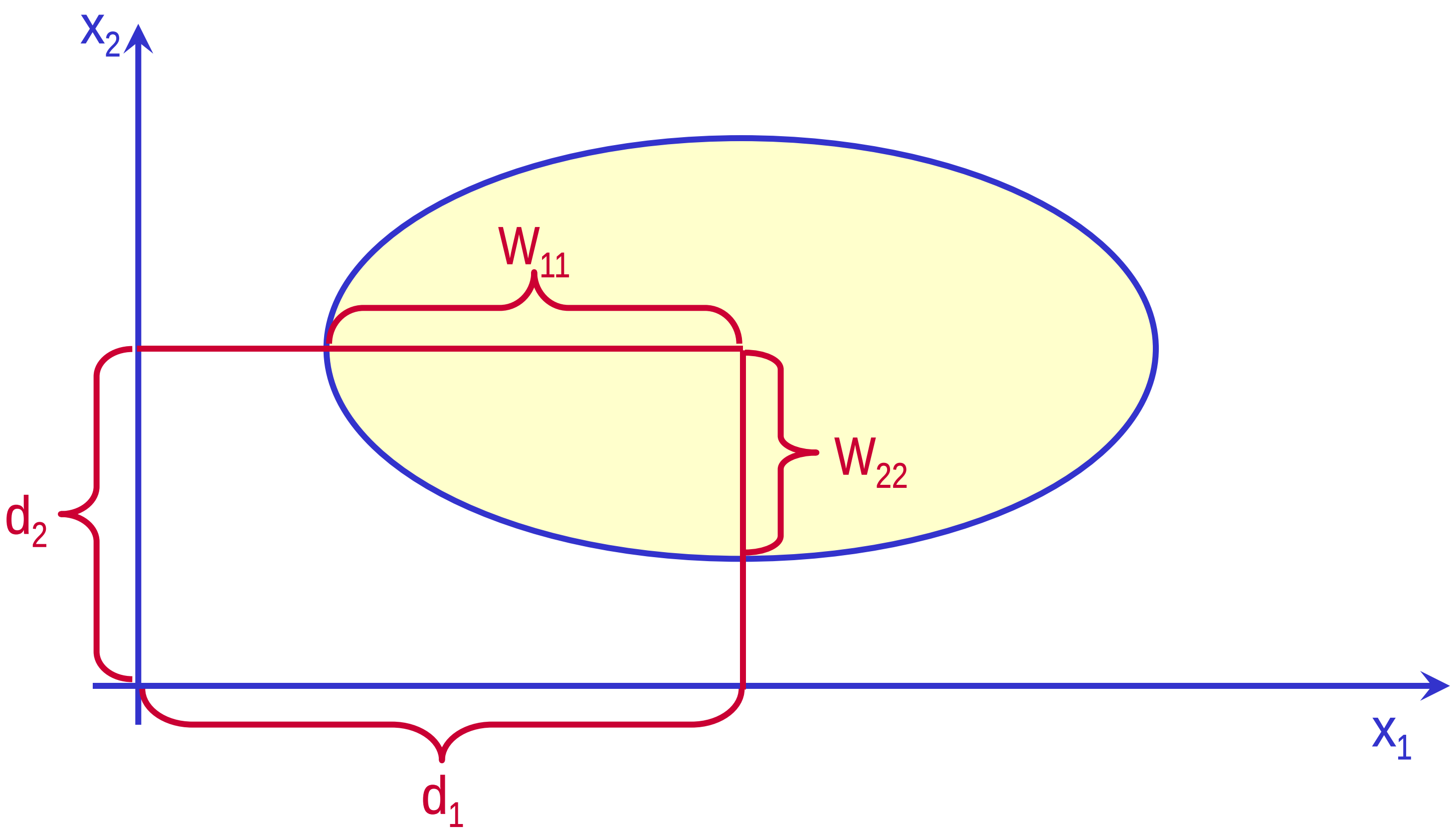
$$\frac{(X_1 - d_1)^2}{W_{11}^2} + \frac{(X_2 - d_2)^2}{W_{22}^2} \le 1$$



Maximum Inscribed Ellipsoid

- If W is diagonal, ellipsoid volume is proportional to

$$W_{11} \cdot W_{22} = \det\left(\begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix}\right)$$



Maximum Inscribed Ellipsoid

- If W is not diagonal, but is symmetric and positive definite

$$X = W \cdot u \quad \left(\|u\|_2 \le 1 \right)$$

We assume d = 0, since d only changes the center of the ellipsoid (not its volume)

$$W \cdot V = V \cdot \Sigma$$
$$V^T V = I$$

(Eigenvalue decomposition)

$$X = V \cdot \Sigma \cdot V^T \cdot u \quad \left(\|u\|_2 \le 1 \right)$$

$$V^T \cdot X = \Sigma \cdot V^T \cdot u \quad \left(\|u\|_2 \le 1 \right)$$

Maximum Inscribed Ellipsoid

- If W is not diagonal, but is symmetric and positive definite

$$V^T \cdot X = \Sigma \cdot V^T \cdot u \quad \left(\|u\|_2 \le 1 \right)$$

Orthogonal rotation

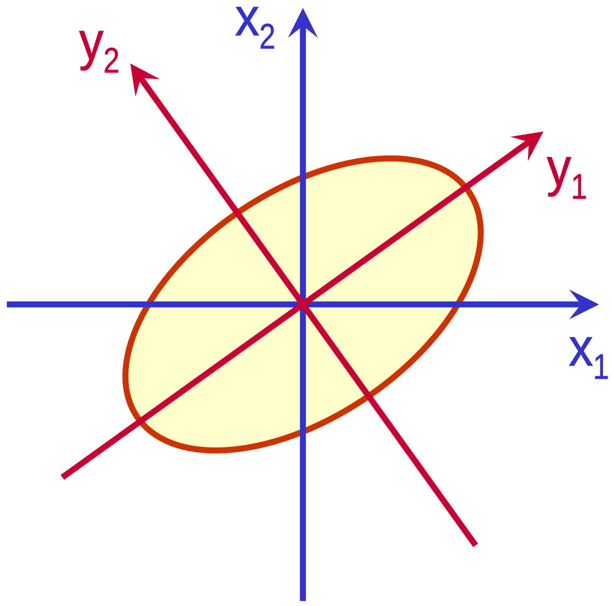
$$Y = V^T \cdot X$$

$$q = V^T \cdot u$$

$$Y = \Sigma \cdot q \quad \left(\|V \cdot q\|_2 \le 1 \right)$$

$$\|Vq\|_2 = \|q\|_2$$

$$Y = \Sigma \cdot q \quad \left(\|q\|_2 \le 1 \right)$$



Orthogonal rotation does not change the ellipsoid volume

Maximum Inscribed Ellipsoid

- If W is not diagonal, but is symmetric and positive definite

$$Y = \Sigma \cdot q \quad \left(\|q\|_2 \leq 1 \right)$$

Σ is diagonal in the new coordinate system

⇓

$$Volume \propto \det(\Sigma)$$

⇓ $W = V \cdot \Sigma \cdot V^T = V \cdot \Sigma \cdot V^{-1}$

$$Volume \propto \det(\Sigma) \cdot \det\left(V \cdot V^{-1}\right) = \det(\Sigma) \cdot \det(V) \cdot \det\left(V^{-1}\right)$$
$$= \det(V) \cdot \det(\Sigma) \cdot \det\left(V^{-1}\right) = \det\left(V \cdot \Sigma \cdot V^{-1}\right) = \det(W)$$

Similarity transformation preserves the determinant value

Maximum Inscribed Ellipsoid

- Inequality constraints

$$\left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \le 0 \quad \left(k = 1,2,\cdots,K\right)$$

- Ellipsoid volume

$$Volume \propto \det\left(W\right)$$

- The maximum inscribed ellipsoid (i.e., W and d) can be founded by solving the following optimization problem

$$\begin{aligned} &\max_{W,d} \quad \det\left(W\right) \\ &S.T. \quad \left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \le 0 \quad \left(k = 1,2,\cdots,K\right) \end{aligned}$$

Do we miss anything?

Maximum Inscribed Ellipsoid

- We need to add the constraint that W must be symmetric and positive definite

$$\max_{W,d} \quad \det(W)$$

$$S.T. \quad \left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \le 0 \quad (k = 1,2,\cdots,K)$$

$$W = W^T$$

$$W \succ 0$$

W is symmetric

W is positive definite

Is this optimization problem easy or difficult to solve?

Maximum Inscribed Ellipsoid

$$
\begin{aligned}
\max_{W,d} \quad & \det(W) \\
S.T. \quad & \left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \le 0 \quad (k = 1, 2, \cdots, K) \\
& W = W^T \\
& W \succ 0
\end{aligned}
$$

- This optimization problem can be converted to a convex programming and it can be solved efficiently

Maximum Inscribed Ellipsoid

$$\begin{aligned} \max_{W,d} \quad & \det(W) \\ S.T. \quad & \left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \leq 0 \quad (k = 1,2,\cdots,K) \\ & W = W^T \\ & W \succ 0 \end{aligned}$$

- Cost function

$$\max \quad \det(W)$$

log(•) is monotonically increasing

$$\max \quad \log[\det(W)]$$

- log[det(W)] is concave – we maximize a concave cost function

Maximum Inscribed Ellipsoid

$$\max_{W,d} \quad \det(W)$$
$$S.T. \quad \left\|W^T B_k\right\|_2 + B_k^T \cdot d + C_k \leq 0 \quad (k = 1,2,\cdots,K)$$
$$W = W^T$$
$$W \succ 0$$

- Constraint

$$\left\|W^T B_k\right\|_2 + B_k^T \cdot d + C_k \leq 0 \quad (k = 1,2,\cdots,K)$$

  - Both $||W^T B_k||_2$ and $B_k^T d$ are convex
  - The sum of two convex functions is convex
  - This constraint set is convex

Maximum Inscribed Ellipsoid

$$\begin{aligned}
\max_{W,d} \quad & \det(W) \\
S.T. \quad & \left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \le 0 \quad (k = 1,2,\cdots,K) \\
& W = W^T \\
& W \succ 0
\end{aligned}$$

- Constraint

$$W = W^T$$

  - Linear equality constraint defines a convex set

Maximum Inscribed Ellipsoid

$$\max_{W,d} \quad \det(W)$$

$$S.T. \quad \left\| W^T B_k \right\|_2 + B_k^T \cdot d + C_k \le 0 \quad (k = 1, 2, \cdots, K)$$

$$W = W^T$$

$$W \succ 0$$

- Constraint

$$W \succ 0$$

  - The set of all positive definite matrices is convex
  - If W_1 and W_2 are positive definite, their positive combination is also positive definite

$$\alpha \cdot W_1 + (1 - \alpha) \cdot W_2 \succ 0$$

Positive coefficients

Maximum Inscribed Ellipsoid

$$\begin{aligned}
\max_{W,d} \quad & \det(W) \\
S.T. \quad & \left\|W^T B_k\right\|_2 + B_k^T \cdot d + C_k \leq 0 \quad (k = 1, 2, \cdots, K) \\
& W = W^T \\
& W \succ 0
\end{aligned}$$

- We maximize a concave function over a convex set – a convex programming problem

- The optimization can be solved by a convex solver
  - E.g., CVX (www.stanford.edu/~boyd/cvx/)
  - We will discuss the optimization algorithm in future lectures...

Summary

- Geometric problems
  - Maximum inscribed ellipsoid
  - Minimum circumscribed ellipsoid