18-660: Numerical Methods for Engineering Design and Optimization

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Electrical & Computer ENGINEERING

Overview

- Compressed Sensing
  - Orthogonal matching pursuit (OMP)

L_1-Norm Regularization

- L_1-norm regularization is often used to find sparse solution of a linear equation

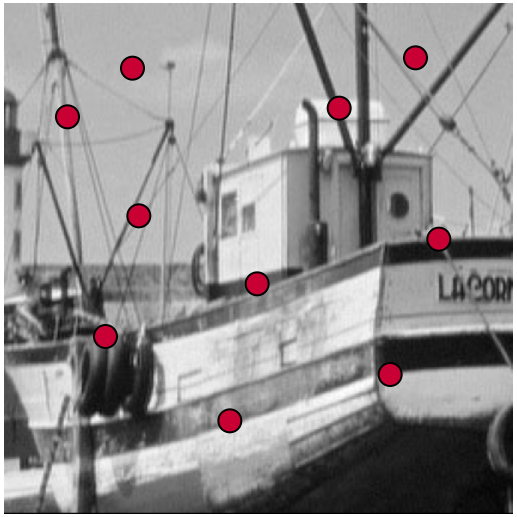
$$\min_{\alpha} \quad \|A\alpha - B\|_2^2$$
$$\text{S.T.} \quad \|\alpha\|_1 \leq \lambda$$

(Convex optimization)

- L_1-norm regularization has been applied to a large number of practical problems
  - In this lecture, we will focus on the compressed sensing problem for image processing

Compressed Sensing

- Our focus: image re-construction
  - Sample an image at a small number of spatial locations
  - Recover full image by a numerical algorithm



Original image

Recovered image

Compressed Sensing

- A 2-D image can be mapped to frequency domain by discrete cosine transform (DCT)

$$\underline{G(u,v)} = \sum_{x=1}^{P}\sum_{y=1}^{Q} a_u \cdot b_v \cdot \underline{g(x,y)} \cdot \cos\frac{\pi(2x-1)(u-1)}{2\cdot P} \cdot \cos\frac{\pi(2y-1)(v-1)}{2\cdot Q}$$

DCT coefficient — Image pixel

Original image

$$x,u \in \{1,2,\cdots,P\}$$

$$y,v \in \{1,2,\cdots,Q\}$$

$$a_u = \begin{cases}\sqrt{1/P} & (u=1)\\ \sqrt{2/P} & (2 \le u \le P)\end{cases}$$

$$b_v = \begin{cases}\sqrt{1/Q} & (v=1)\\ \sqrt{2/Q} & (2 \le v \le Q)\end{cases}$$

Compressed Sensing

- A 2-D image can be uniquely determined by inverse discrete cosine transform (IDCT), if all DCT coefficients are known

$$g(x,y)=\sum_{u=1}^{P}\sum_{v=1}^{Q}a_u\cdot b_v\cdot G(u,v)\cdot\cos\frac{\pi(2x-1)(u-1)}{2\cdot P}\cdot\cos\frac{\pi(2y-1)(v-1)}{2\cdot Q}$$

Image pixel — DCT coefficient

Original image

$$x,u\in\{1,2,\cdots,P\}$$

$$y,v\in\{1,2,\cdots,Q\}$$

$$a_u=\begin{cases}\sqrt{1/P} & (u=1)\\ \sqrt{2/P} & (2\le u\le P)\end{cases}$$

$$b_v=\begin{cases}\sqrt{1/Q} & (v=1)\\ \sqrt{2/Q} & (2\le v\le Q)\end{cases}$$

Compressed Sensing

- Sample a 2-D image at a number of (say, M) spatial locations
  - Result in a set of (i.e., M) linear equations

$$g(x_1,y_1)=\sum_{u=1}^{P}\sum_{v=1}^{Q}a_u\cdot b_v\cdot G(u,v)\cdot\cos\frac{\pi(2x_1-1)(u-1)}{2\cdot P}\cdot\cos\frac{\pi(2y_1-1)(v-1)}{2\cdot Q}$$

$$g(x_2,y_2)=\sum_{u=1}^{P}\sum_{v=1}^{Q}a_u\cdot b_v\cdot G(u,v)\cdot\cos\frac{\pi(2x_2-1)(u-1)}{2\cdot P}\cdot\cos\frac{\pi(2y_2-1)(v-1)}{2\cdot Q}$$

$$\vdots$$

$$\underline{g(x_M,y_M)}=\sum_{u=1}^{P}\sum_{v=1}^{Q}a_u\cdot b_v\cdot \underline{G(u,v)}\cdot\cos\frac{\pi(2x_M-1)(u-1)}{2\cdot P}\cdot\cos\frac{\pi(2y_M-1)(v-1)}{2\cdot Q}$$

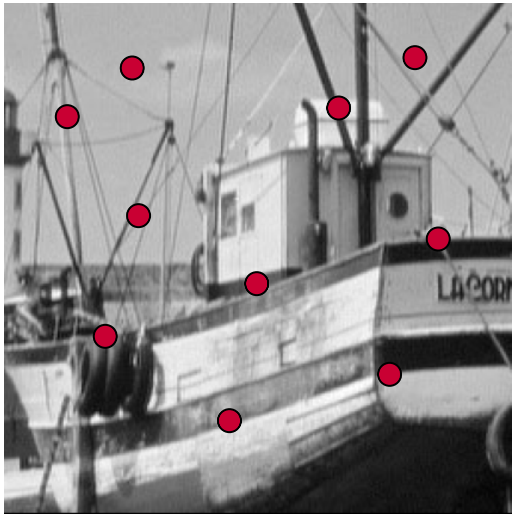
Image pixel

DCT coefficient

Our goal is to solve all DCT coefficients from these linear equations

Compressed Sensing

- Re-write the linear equation in matrix form



Original image

IDCT transform

$$\begin{bmatrix} \times \\ \times \\ \times \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \cdot \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}$$

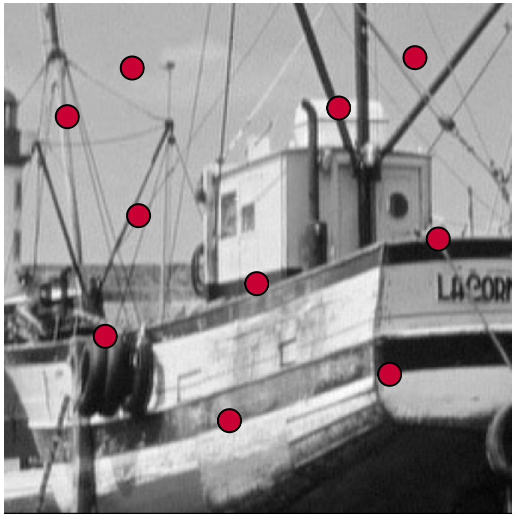
Sampling data

DCT coefficients

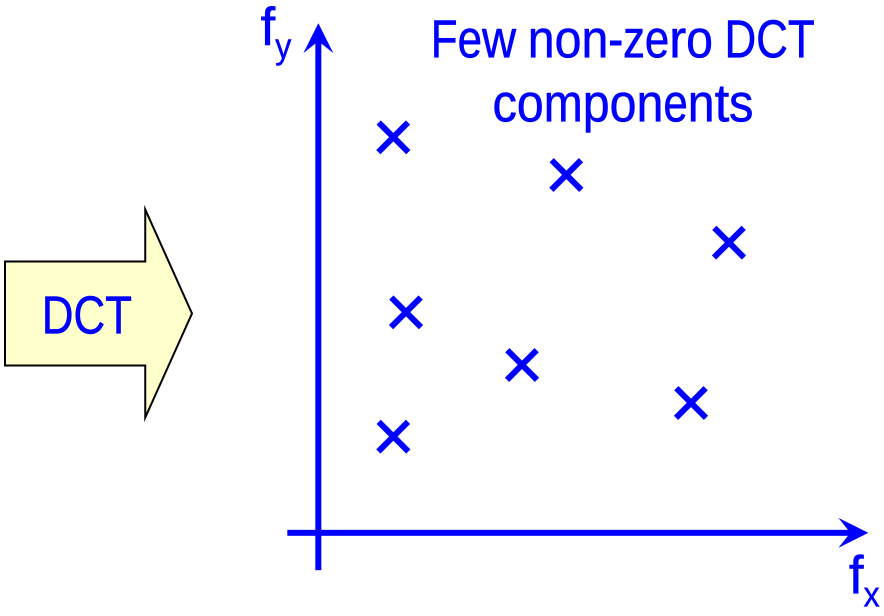
Result in an under-determined linear equation, since we have less sampling locations than unknown DCT coefficients

Compressed Sensing

- Additional information is required to uniquely solve under-determined linear equation



Original image



Explore sparsity to find unique, deterministic solution from under-determined equation

Compressed Sensing

- We know that **α** is sparse – but we do *not* know the exact location of zeros
  - Find the sparse solution α (i.e., DCT coefficients) for $A\alpha = B$
  - Apply inverse DCT transform to recover full image

$$\underbrace{\begin{bmatrix} \times \\ \times \\ \times \end{bmatrix}}_{B} = \overbrace{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}}^{A} \cdot \underbrace{\begin{bmatrix} 0 \\ \times \\ \times \\ 0 \\ 0 \end{bmatrix}}_{\alpha \text{ (Sparse)}}$$

$$\min_{\alpha} \quad \|A\alpha - B\|_2^2$$
$$\text{S.T.} \quad \|\alpha\|_1 \le \lambda$$

(Convex optimization)

Compressed Sensing

- Compressed sensing is a general technique that is applicable to many other practical problems
  - Image re-construction is one of the application examples

- More details on compressed sensing can be found at

Candes, “Compressive sampling,” International Congress of Mathematicians, 2006

Donoho, “Compressed sensing,” IEEE Trans. Information Theory, 2006.

Orthogonal Matching Pursuit (OMP)

- Another way to find sparse solution is L_0-norm regularization

$$\min_{\alpha} \quad \|A\alpha - B\|_2^2$$
$$\text{S.T.} \quad \|\alpha\|_0 \leq \lambda$$

(VERY difficult to solve)

Number of non-zeros in α

- Efficient numerical algorithm exists to find an approximate (i.e., sub-optimal) solution
  - E.g., orthogonal matching pursuit

Orthogonal Matching Pursuit (OMP)

- Goal:
  - Identify a subset of DCT coefficients that are non-zero
- Approach:
  - Find important DCT coefficients by checking the inner product between A_i and B
  - Assume that each A_i is normalized (i.e., has unit length)

$$\underbrace{\begin{bmatrix} \times \\ \times \\ \times \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} A_1 & A_2 & A_3 & A_4 & \cdots \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{bmatrix}}_{\alpha}$$

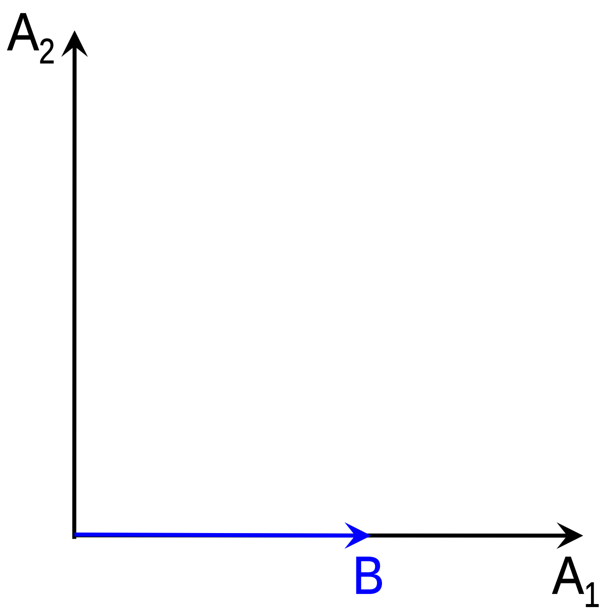
$$\langle A_i, B \rangle = A_i^T B$$

Inner product

Orthogonal Matching Pursuit (OMP)

- Inner product <A_i, B> implies the importance of A_i when approximating B

- 2-D example



$$\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 \approx B$$

$$\langle A_1, B \rangle \neq 0$$
$$\langle A_2, B \rangle = 0$$

A_1 is important to approximate B

Orthogonal Matching Pursuit (OMP)

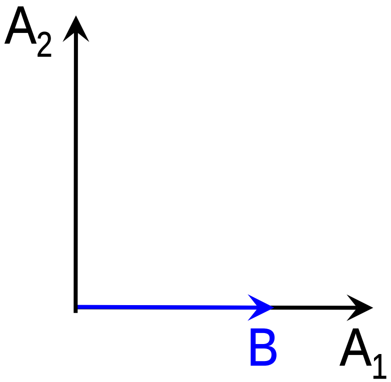
- 2-D example

$$\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 \approx B$$

  - Step 1: Calculate <A_1, B> and <A_2, B>
  - Step 2: Select A_i that corresponds to the largest inner product magnitude (i.e., A_1 in this example)
  - Step 3: Solve the coefficient α_1 by least-squares fitting

$$\min_{\alpha_1} \quad \left\| \alpha_1 \cdot A_1 - B \right\|_2^2$$

  - Step 4: Set α_2 = 0 (B is independent of A_2 in this example)

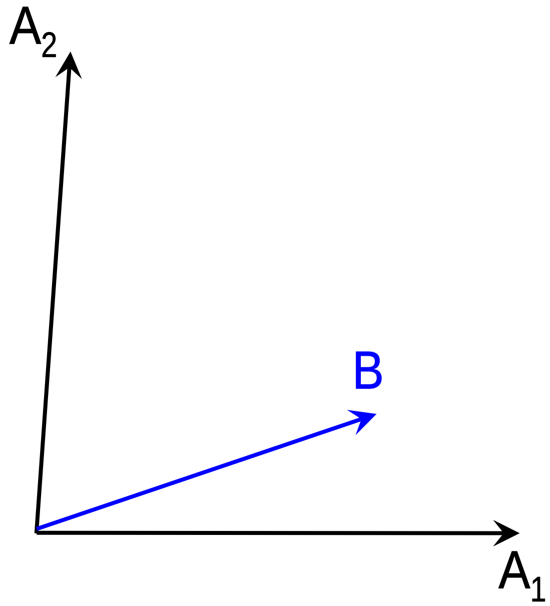


Orthogonal Matching Pursuit (OMP)

- 2-D example

$$\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 \approx B$$

  - A_1 and A_2 are not orthogonal
  - B is not orthogonal to A_1 or A_2



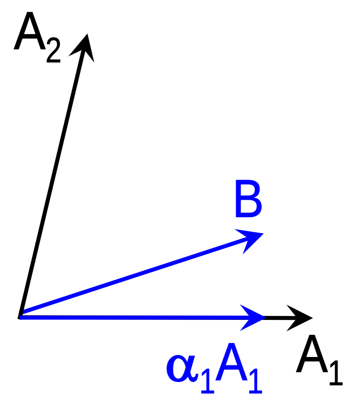
Orthogonal Matching Pursuit (OMP)

- 2-D example

$$\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 \approx B$$

  - Step 1: Calculate <A_1, B> and <A_2, B>
  - Step 2: Select A_i that corresponds to the largest inner product magnitude (i.e., A_1 in this example)
  - Step 3: Solve the coefficient α_1 by least-squares fitting

$$\min_{\alpha_1} \quad \left\| \alpha_1 \cdot A_1 - B \right\|_2^2$$



Orthogonal Matching Pursuit (OMP)

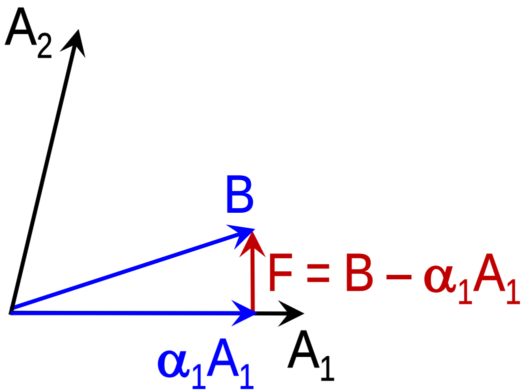
- 2-D example

$$\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 \approx B$$

  - Step 4: Calculate the residue

$$F = B - \alpha_1 \cdot A_1$$ (F is orthogonal to A_1)

  - Step 5: Calculate <A_1, F> and <A_2, F>
  - Step 6: Select A_i that corresponds to the largest inner product magnitude (i.e., A_2 in this example)



Orthogonal Matching Pursuit (OMP)

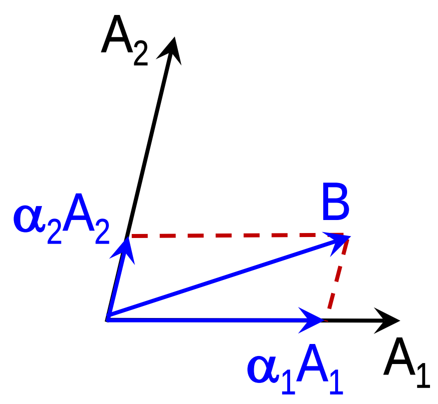
- 2-D example

$$\alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 \approx B$$

  - Step 7: Solve the coefficients α_1 and α_2 by least-squares fitting

$$\min_{\alpha_1, \alpha_2} \quad \left\| \alpha_1 \cdot A_1 + \alpha_2 \cdot A_2 - B \right\|_2^2$$

(α_1 is re-calculated)



Orthogonal Matching Pursuit (OMP)

- General OMP algorithm to solve underdetermined linear equation

$$\min_{\alpha} \quad \|A\alpha - B\|_2^2$$
$$\text{S.T.} \quad \|\alpha\|_0 \le \lambda$$

  - Step 1: Set F = B, $\Omega = \{\ \}$ and p = 1
  - Step 2: Calculate the inner product values θ_i = <A_i, F>
  - Step 3: Identify the index s for which $|\theta_s|$ takes the largest value
  - Step 4: Update Ω by $\Omega = \Omega \cup \{s\}$
  - Step 5: Approximate B by the linear combination of $\{A_i;\ i \in \Omega\}$

$$\min_{\alpha_i, i\in\Omega} \quad \left\| \sum_{i\in\Omega} \alpha_i \cdot A_i - B \right\|_2^2$$

  - Step 6: Update F

$$F = B - \sum_{i\in\Omega} \alpha_i \cdot A_i$$

  - Step 7: If $p < \lambda$, p = p+1 and go to Step 2. Otherwise, stop.

$$\alpha_i = 0 \quad (i \notin \Omega)$$

Orthogonal Matching Pursuit (OMP)

$$\min_{\alpha} \quad \left\| A\alpha - B \right\|_2^2$$
$$\text{S.T.} \quad \left\| \alpha \right\|_0 \le \lambda$$

- Orthogonal matching pursuit is a heuristic algorithm to solve the L_0-norm regularization problem

- A number of other heuristic algorithms exist to solve under-determined linear equation
  - More details can be found in compressed sensing papers

Summary

- Compressed sensing
  - Orthogonal matching pursuit (OMP)