18-660: Numerical Methods for Engineering Design and Optimization

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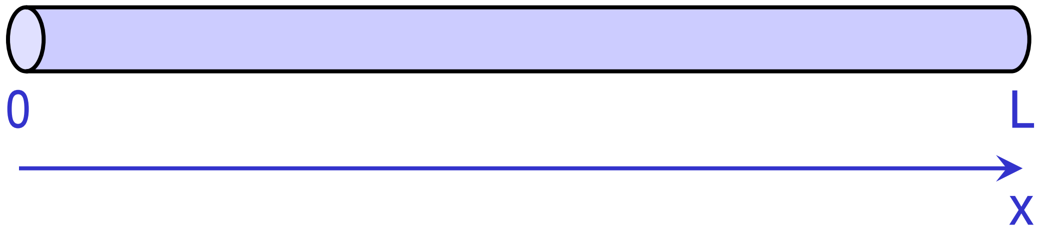
Electrical & Computer ENGINEERING

Overview

- Thermal Analysis
  - 2-D / 3-D heat equation
  - Finite difference

1-D Heat Equation

- The complete PDE with boundary and initial conditions



PDE $$\rho \cdot C_p \cdot T_t(x,t) = \kappa \cdot T_{xx}(x,t) + f(x,t) \quad (0 \le x \le L \quad 0 \le t \le \infty)$$

BCs $$\begin{cases} T(x=0,t) = T_1 \\ T(x=L,t) = T_2 \end{cases} \quad (0 < t \le \infty)$$

IC $$T(x,t=0) = T_0 \quad (0 \le x \le L)$$

2-D / 3-D Heat Equation

- 2-D heat equation

$$\rho \cdot C_p \cdot \frac{\partial T(x,y,t)}{\partial t} = \kappa \cdot \nabla^2 T(x,y,t) + f(x,y,t)$$

- 3-D heat equation

Density

Laplace operator

$$\rho \cdot C_p \cdot \frac{\partial T(x,y,z,t)}{\partial t} = \kappa \cdot \nabla^2 T(x,y,z,t) + f(x,y,z,t)$$

Thermal capacity

Thermal conductivity

Heat source

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2-D / 3-D Heat Equation

- Heat equation is a 2nd-order linear PDE

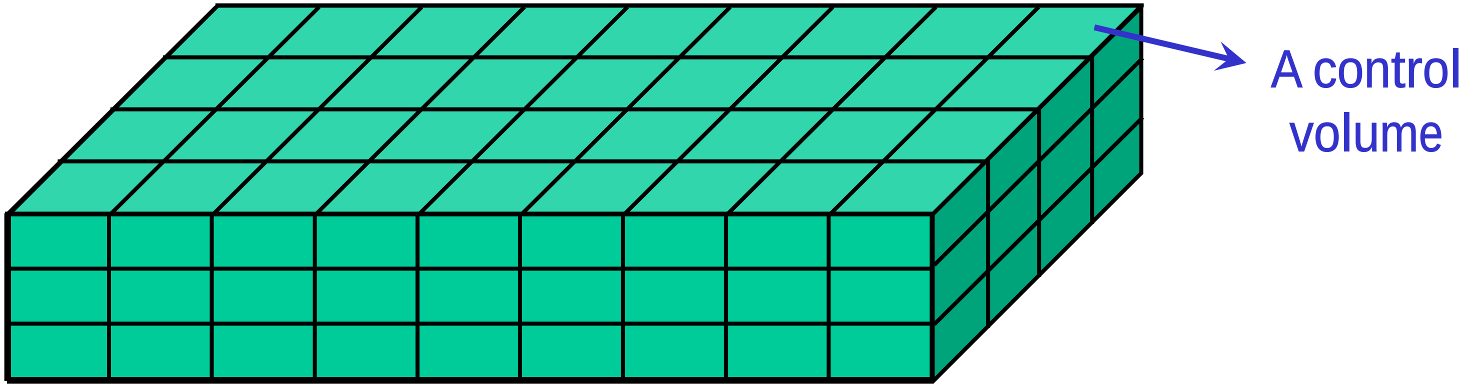
$$\rho \cdot C_p \cdot \frac{\partial T(x,y,z,t)}{\partial t} = \kappa \cdot \nabla^2 T(x,y,z,t) + f(x,y,z,t)$$

- Order of PDE – the order of the highest partial derivative
- Linearity – the dependent variable T and all its derivatives appear in a linear fashion
- Homogeneity
  - Homogenous if $f(x,y,z,t) = 0$
  - Non-homogenous if $f(x,y,z,t) \neq 0$

Finite Difference Method

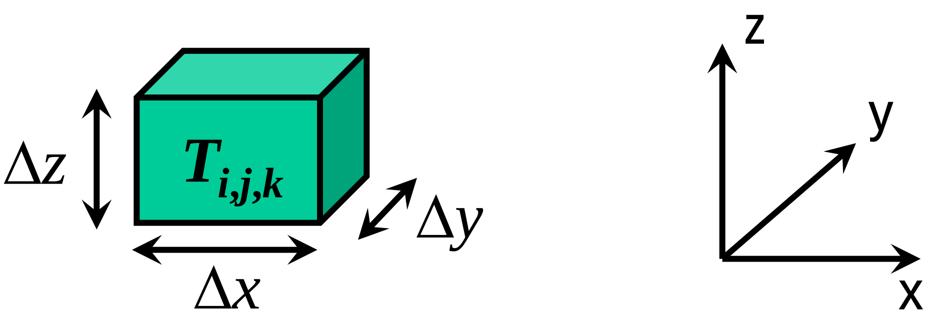
- PDE can be numerically solved using finite difference method
  - Discretize 3-D space into a number of small control volumes

$$\rho \cdot C_p \cdot \frac{\partial T(x,y,z,t)}{\partial t} = \kappa \cdot \nabla^2 T(x,y,z,t) + f(x,y,z,t)$$



Finite Difference Method

- A control volume



- Write PDE for each control volume

$$\rho \cdot C_p \cdot \frac{\partial T(i,j,k,t)}{\partial t} = \kappa \cdot \left[\frac{\partial^2 T(i,j,k,t)}{\partial x^2} + \frac{\partial^2 T(i,j,k,t)}{\partial y^2} + \frac{\partial^2 T(i,j,k,t)}{\partial z^2} \right] + f(i,j,k,t)$$

Finite Difference Method

- Discretize PDE over the control volume

$$\frac{\partial^2 T(i,j,k,t)}{\partial x^2}$$

$$\Downarrow$$

$$\frac{\dfrac{\partial T(i+1,j,k,t)}{\partial x} - \dfrac{\partial T(i,j,k,t)}{\partial x}}{\Delta x}$$

$$\Downarrow$$

$$\frac{\dfrac{[T(i+1,j,k,t) - T(i,j,k,t)]}{\Delta x} - \dfrac{[T(i,j,k,t) - T(i-1,j,k,t)]}{\Delta x}}{\Delta x}$$

$$\Downarrow$$

$$\frac{T_{i+1,j,k} - T_{i,j,k}}{(\Delta x)^2} - \frac{T_{i,j,k} - T_{i-1,j,k}}{(\Delta x)^2}$$

Finite Difference Method

- Rewrite the finite difference discretization

$$\rho \cdot C_p \cdot \frac{\partial T(i,j,k,t)}{\partial t} = \kappa \cdot \left[\frac{\partial^2 T(i,j,k,t)}{\partial x^2} + \frac{\partial^2 T(i,j,k,t)}{\partial y^2} + \frac{\partial^2 T(i,j,k,t)}{\partial z^2} \right] + f(i,j,k,t)$$

$$\Downarrow$$

$$\rho \cdot C_p \cdot \frac{\partial T_{i,j,k}}{\partial t} = f_{i,j,k} + \frac{\kappa \cdot \left[T_{i+1,j,k} - T_{i,j,k}\right]}{(\Delta x)^2} - \frac{\kappa \cdot \left[T_{i,j,k} - T_{i-1,j,k}\right]}{(\Delta x)^2} +$$
$$\frac{\kappa \cdot \left[T_{i,j+1,k} - T_{i,j,k}\right]}{(\Delta y)^2} - \frac{\kappa \cdot \left[T_{i,j,k} - T_{i,j-1,k}\right]}{(\Delta y)^2} + \frac{\kappa \cdot \left[T_{i,j,k+1} - T_{i,j,k}\right]}{(\Delta z)^2} - \frac{\kappa \cdot \left[T_{i,j,k} - T_{i,j,k-1}\right]}{(\Delta z)^2}$$

Finite Difference Method

$$\rho \cdot C_p \cdot \frac{\partial T_{i,j,k}}{\partial t} = f_{i,j,k} + \frac{\kappa \cdot \left[T_{i+1,j,k} - T_{i,j,k}\right]}{(\Delta x)^2} - \frac{\kappa \cdot \left[T_{i,j,k} - T_{i-1,j,k}\right]}{(\Delta x)^2} +$$

$$\frac{\kappa \cdot \left[T_{i,j+1,k} - T_{i,j,k}\right]}{(\Delta y)^2} - \frac{\kappa \cdot \left[T_{i,j,k} - T_{i,j-1,k}\right]}{(\Delta y)^2} + \frac{\kappa \cdot \left[T_{i,j,k+1} - T_{i,j,k}\right]}{(\Delta z)^2} - \frac{\kappa \cdot \left[T_{i,j,k} - T_{i,j,k-1}\right]}{(\Delta z)^2}$$

$I_{i,j,k}$ $\quad$ C $\quad$ G_x $\quad$ G_y $\quad$ G_z

$$\boxed{f_{i,j,k} \cdot \Delta x \cdot \Delta y \cdot \Delta z} = \boxed{\rho \cdot C_p \cdot \Delta x \cdot \Delta y \cdot \Delta z} \cdot \frac{\partial T_{i,j,k}}{\partial t} +$$

$$\boxed{\frac{\kappa \cdot \Delta y \cdot \Delta z}{\Delta x}} \cdot \left[T_{i,j,k} - T_{i+1,j,k}\right] + \frac{\kappa \cdot \Delta y \cdot \Delta z}{\Delta x} \cdot \left[T_{i,j,k} - T_{i-1,j,k}\right] +$$

$$\boxed{\frac{\kappa \cdot \Delta x \cdot \Delta z}{\Delta y}} \cdot \left[T_{i,j,k} - T_{i,j+1,k}\right] + \frac{\kappa \cdot \Delta x \cdot \Delta z}{\Delta y} \cdot \left[T_{i,j,k} - T_{i,j-1,k}\right] +$$

$$\boxed{\frac{\kappa \cdot \Delta x \cdot \Delta y}{\Delta z}} \cdot \left[T_{i,j,k} - T_{i,j,k+1}\right] + \frac{\kappa \cdot \Delta x \cdot \Delta y}{\Delta z} \cdot \left[T_{i,j,k} - T_{i,j,k-1}\right]$$

Finite Difference Method

- We have:

$$I_{i,j,k} = C \cdot \frac{\partial T_{i,j,k}}{\partial t} + G_x \cdot \left[T_{i,j,k} - T_{i+1,j,k}\right] + G_x \cdot \left[T_{i,j,k} - T_{i-1,j,k}\right]$$

$$G_y \cdot \left[T_{i,j,k} - T_{i,j+1,k}\right] + G_y \cdot \left[T_{i,j,k} - T_{i,j-1,k}\right] + G_z \cdot \left[T_{i,j,k} - T_{i,j,k+1}\right] + G_z \cdot \left[T_{i,j,k} - T_{i,j,k-1}\right]$$

  - where

$$G_x = \frac{\kappa \cdot \Delta y \cdot \Delta z}{\Delta x} \quad G_y = \frac{\kappa \cdot \Delta x \cdot \Delta z}{\Delta y} \quad G_z = \frac{\kappa \cdot \Delta x \cdot \Delta y}{\Delta z}$$

$$C = \rho \cdot C_p \cdot \Delta x \cdot \Delta y \cdot \Delta z \quad I_{i,j,k} = f_{i,j,k} \cdot \Delta x \cdot \Delta y \cdot \Delta z$$

The discretized thermal equation has a form similar to a circuit equation

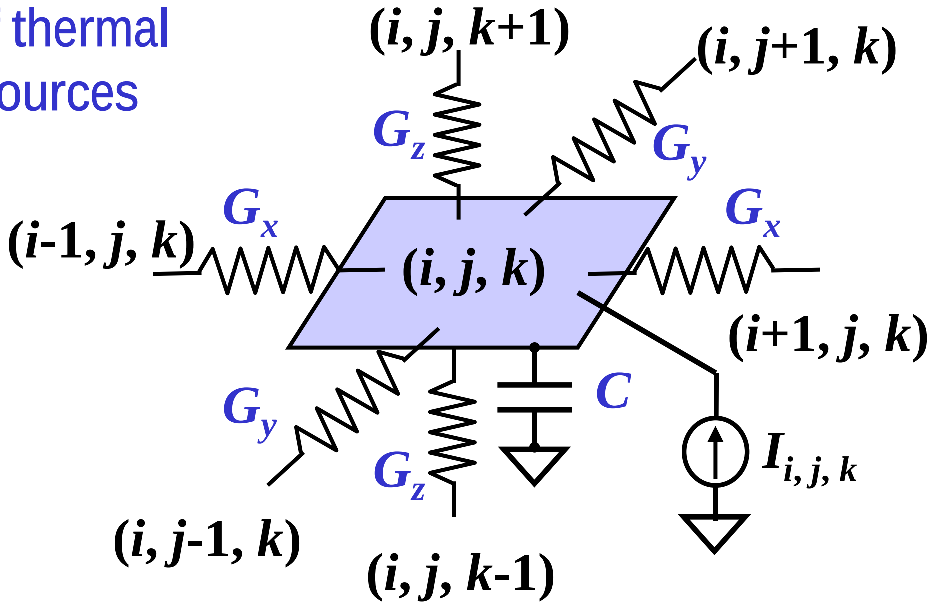
Finite Difference Method

$$I_{i,j,k} = C \cdot \frac{\partial T_{i,j,k}}{\partial t} + G_x \cdot \left[T_{i,j,k} - T_{i+1,j,k}\right] + G_x \cdot \left[T_{i,j,k} - T_{i-1,j,k}\right]$$
$$G_y \cdot \left[T_{i,j,k} - T_{i,j+1,k}\right] + G_y \cdot \left[T_{i,j,k} - T_{i,j-1,k}\right] + G_z \cdot \left[T_{i,j,k} - T_{i,j,k+1}\right] + G_z \cdot \left[T_{i,j,k} - T_{i,j,k-1}\right]$$

Equivalent circuit consisting of thermal resistors/capacitors and heat sources

T == nodal voltage

I == branch current



Finite Difference Method

$$I_{i,j,k} = C \cdot \frac{\partial T_{i,j,k}}{\partial t} + G_x \cdot \left[T_{i,j,k} - T_{i+1,j,k}\right] + G_x \cdot \left[T_{i,j,k} - T_{i-1,j,k}\right]$$
$$G_y \cdot \left[T_{i,j,k} - T_{i,j+1,k}\right] + G_y \cdot \left[T_{i,j,k} - T_{i,j-1,k}\right] + G_z \cdot \left[T_{i,j,k} - T_{i,j,k+1}\right] + G_z \cdot \left[T_{i,j,k} - T_{i,j,k-1}\right]$$

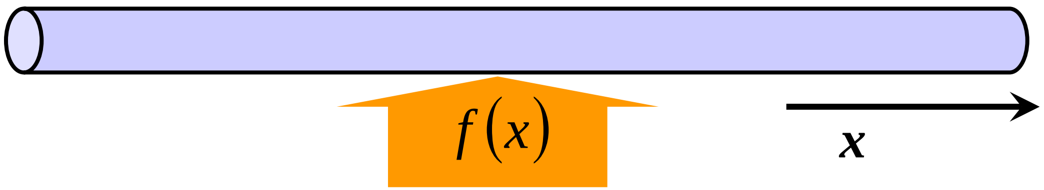
- The operator ∂/∂t can be handled by numerical integration
  - We need to solve a large-scale linear equation to find $T_{i,j,k}(t_n)$ at each time point t_n
- Generally interested only in steady state – thermal capacitance is not considered

1-D Thermal Analysis Example

- 1-D PDE to describe the steady-state temperature distribution along a uniform rod at [0, 1]

$$T(x,0) = T_{Init}$$

$$T(x=0,t) = T(x=1,t) = 0$$



$$\rho \cdot C_p \cdot \frac{\partial T(x,t)}{\partial t} = \kappa \cdot \frac{\partial T^2(x,t)}{\partial x^2} + f(x,t)$$

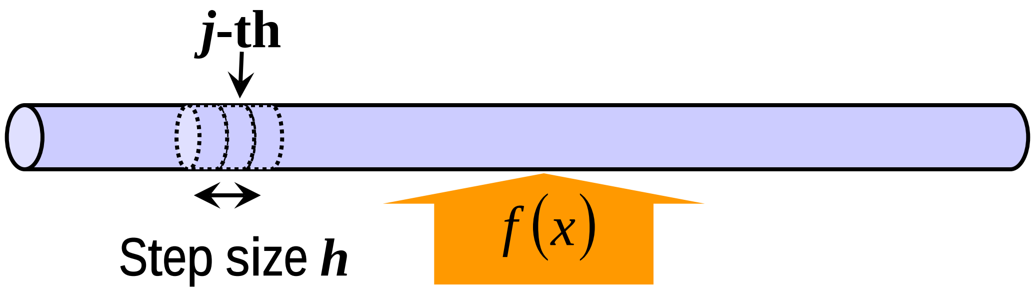
Steady state

$$\frac{\partial T(x,t)}{\partial t} = 0$$

$$-\kappa \cdot T_{xx}(x) = f(x) \quad (0 < x < 1)$$

1-D Thermal Analysis Example

- Approximate 2nd order derivative using finite difference



$$T_j = T(x_j)$$
$$f_j = f(x_j)$$

$$T(x=0,t) = T(x=1,t) = 0 \qquad -\kappa \cdot T_{xx}(x) = f(x) \qquad T_{xx}(x_j) = \frac{T_{j+1} + T_{j-1} - 2T_j}{h^2}$$

$$\kappa \cdot \left(-T_{j-1} + 2T_j - T_{j+1}\right) = h^2 \cdot f_j \quad (1 \le j \le N-1)$$
$$T_0 = T_N = 0$$

1-D Thermal Analysis Example

- The linear system is:

$$\kappa \cdot \left(-T_{j-1} + 2T_j - T_{j+1}\right) = h^2 \cdot f_j \quad \left(1 \le j \le N-1\right)$$

$$T_0 = T_N = 0$$

$$\Downarrow$$

$$\kappa \cdot \left(-0 + 2T_1 - T_2\right) = h^2 \cdot f_1$$

$$\kappa \cdot \left(-T_1 + 2T_2 - T_3\right) = h^2 \cdot f_2$$

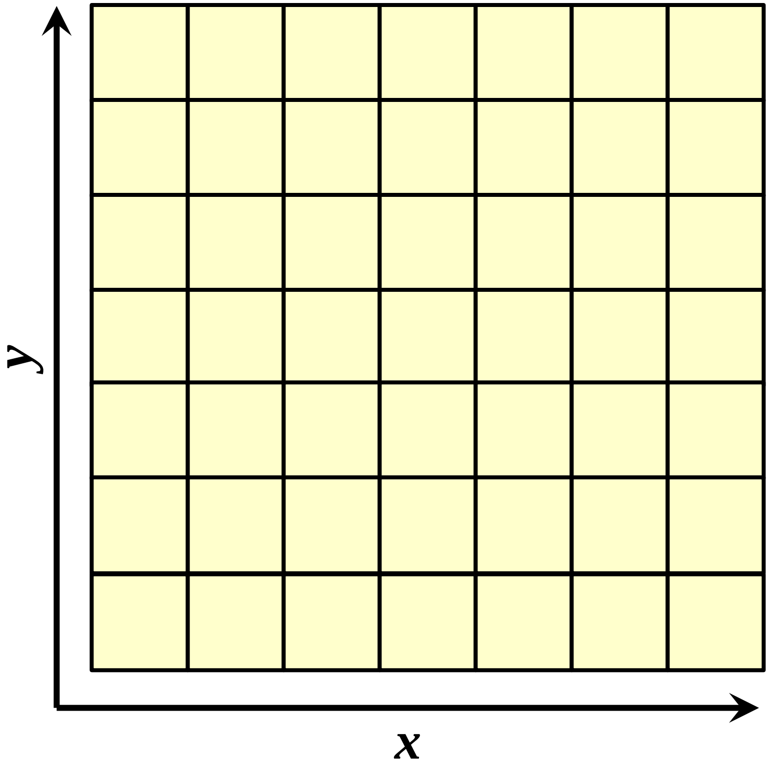
$$\vdots$$

$$\kappa \cdot \left(-T_{N-2} + 2T_{N-1} - 0\right) = h^2 \cdot f_{N-1}$$

Solve linear equation to determine T_1, T_2, ..., T_{N-1}

2-D Thermal Analysis Example

- 2-D PDE to describe the steady-state temperature distribution over a uniform plane x, y ∈ [0, 1]



$$T(x,y,0) = T_{Init}$$

$$T(x=0,t) = T(x=1,t) = 0$$

$$T(y=0,t) = T(y=1,t) = 0$$

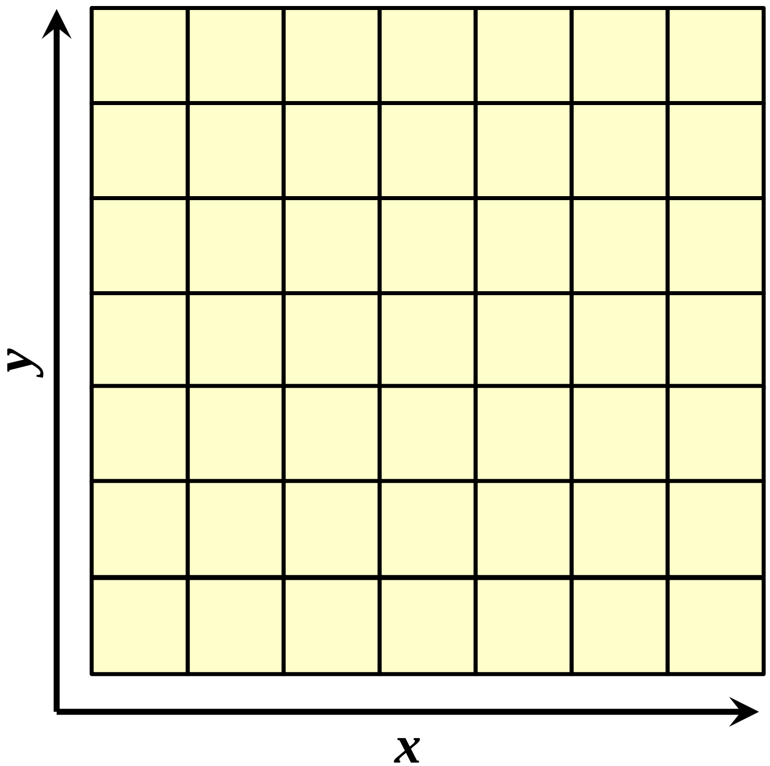
$$\rho \cdot C_p \cdot \frac{\partial T}{\partial t} = \kappa \cdot \left(\frac{\partial T^2}{\partial x^2} + \frac{\partial T^2}{\partial y^2} \right) + f(x,y,t)$$

Steady state ⇩ $\frac{\partial T}{\partial t} = 0$

$$-\kappa \cdot (T_{xx} + T_{yy}) = f(x,y) \quad (0 < x, y < 1)$$

2-D Thermal Analysis Example

- Approximate 2nd order derivative using finite difference



$$T(x=0,t)=T(x=1,t)=0$$
$$T(y=0,t)=T(y=1,t)=0$$
$$-\kappa \cdot (T_{xx}+T_{yy}) = f(x,y)$$

$$T_{i,j} = T(x_i, y_j)$$
$$f_{i,j} = f(x_i, y_j)$$

$$T_{xx}(x_i, y_j) = \frac{T_{i+1,j}+T_{i-1,j}-2T_{i,j}}{h^2}$$
$$T_{yy}(x_i, y_j) = \frac{T_{i,j+1}+T_{i,j-1}-2T_{i,j}}{h^2}$$

2-D Thermal Analysis Example

- The linear system is:

$$T(x=0,t)=T(x=1,t)=0$$
$$T(y=0,t)=T(y=1,t)=0$$
$$-\kappa\cdot\left(T_{xx}+T_{yy}\right)=f(x,y)$$

$$T_{xx}\left(x_i,y_j\right)=\frac{T_{i+1,j}+T_{i-1,j}-2T_{i,j}}{h^2}$$

$$T_{yy}\left(x_i,y_j\right)=\frac{T_{i,j+1}+T_{i,j-1}-2T_{i,j}}{h^2}$$

⇩

$$\kappa\cdot\left(-T_{i-1,j}+2T_{i,j}-T_{i+1,j}-T_{i,j-1}+2T_{i,j}-T_{i,j+1}\right)=h^2\cdot f_{i,j}$$ $T_{i,j}$ NOT at boundary

$$T_{i,j}=0$$ $T_{i,j}$ at boundary

Solve linear equation to determine all temperature values

Thermal Analysis

- Thermal analysis generally requires to solve a large-scale linear equation

$$A \cdot X = B$$

- The matrix A is symmetric and diagonally dominant

$$A_{ij} = A_{ji}$$ For all matrix elements

$$\left|A_{ii}\right| \geq \sum_{j=1,\, j\neq i}^{N} \left|A_{ij}\right|$$ For all diagonal elements

Thermal Analysis

- 1-D thermal analysis example

$$\kappa \cdot \left(-T_{j-1} + 2T_j - T_{j+1}\right) = h^2 \cdot f_j \quad \left(1 \le j \le N-1\right)$$

$$T_0 = T_N = 0$$

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ \vdots \\ \vdots \\ T_{N-1} \end{bmatrix} = \frac{h^2}{\kappa} \cdot \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{N-1} \end{bmatrix}$$

Symmetric and
diagonally dominant

Thermal Analysis

- A matrix "A" is positive definite, if
  - A is symmetric *and*
  - A is diagonally dominant *and*
  - All diagonal elements of A are non-negative *and*
  - A is not singular
  - *Sufficient* but *NOT necessary* condition

- Definition of positive definite matrix

$$P^T \cdot A \cdot P > 0$$ for any real-valued vector $P \neq 0$

All eigenvalues of A are positive

Thermal Analysis

- Positive definite linear equation AX = B can be solved by efficient numerical algorithms
  - Cholesky decomposition
  - Conjugate gradient method
  - Etc.

- We will try to cover some of these efficient algorithms in future lectures

Summary

- Thermal analysis
  - 2-D / 3-D heat equation
  - Finite difference