18-660: Numerical Methods for Engineering Design and Optimization

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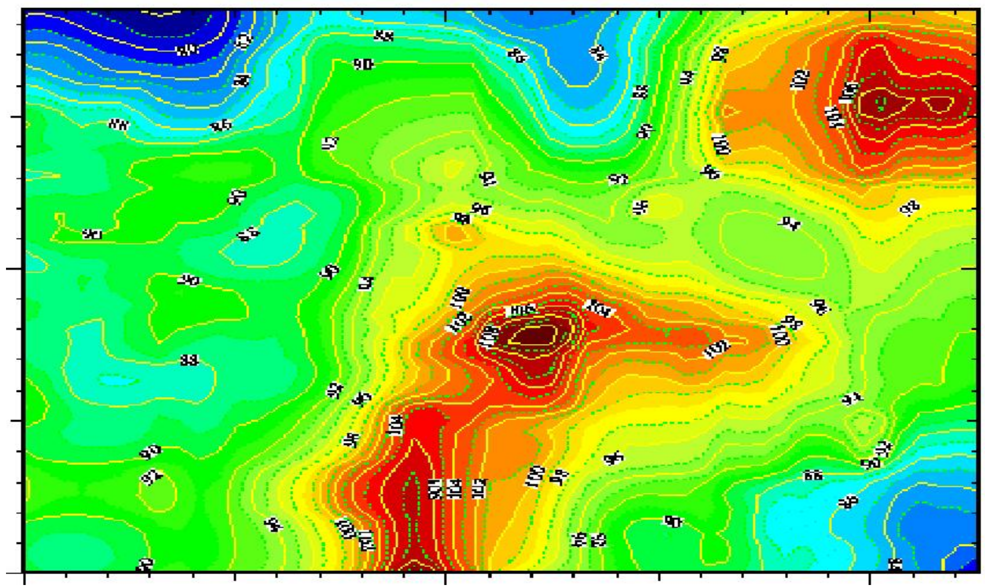
Electrical & Computer ENGINEERING

Overview

- Partial Differential Equation (PDE)
  - Heat equation
  - Boundary condition

Partial Differential Equation (PDE)

- Partial differential equation is often used to describe a physical system or process
- Example: heat equation is an PDE for thermal analysis
  - We will derive heat equation step by step in this lecture



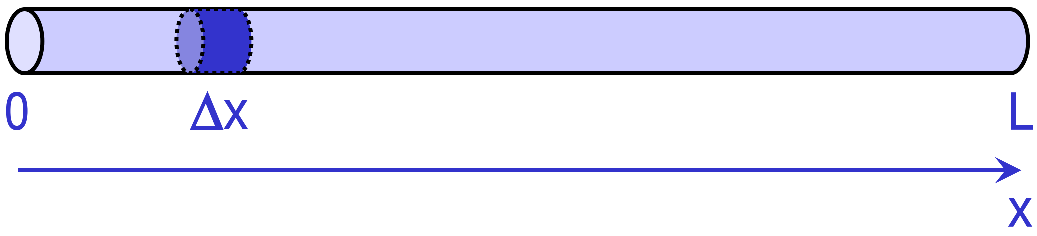
Heat Equation

- Heat equation can be derived from several fundamental physics laws
  - Fourier's law
  - Conservation of heat
  - Etc.

- We will use a 1-D example to illustrate heat equation
  - Help to get many insights about heat transfer process

1-D Heat Model

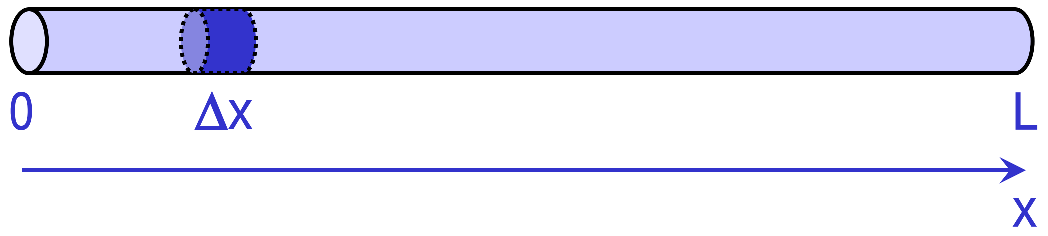
- We consider a 1-D rod of length L



- Three major assumptions
  - Rod is made of a single homogenous conducting material
  - Rod is laterally insulated (heat flows only in the x-direction)
  - Rod is thin (constant temperature at all points of a cross section)

Conservation of Heat

- Apply conservation of heat to the segment [x, x+Δx]

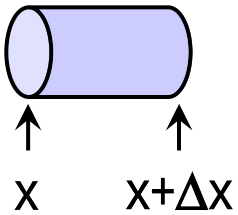


Net change of heat inside [x, x+Δx] = Net flux of heat across boundaries + Total heat generated inside [x, x+Δx]

We will look at each of these three components in detail

Conservation of Heat

Net change of heat inside [x, x+Δx] = Net flux of heat across boundaries + Total heat generated inside [x, x+Δx]



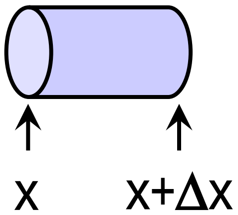
- Total heat inside [x, x+Δx] is equal to:

$$\int_{x}^{x+\Delta x} \rho \cdot C_p \cdot A \cdot T(s,t) \cdot ds$$

ρ:	density
C_p:	thermal capacity (measure the ability to store heat)
A:	cross-section area
T:	temperature
t:	time
s:	x-coordinate

Conservation of Heat

Net change of heat inside [x, x+Δx] $=$ Net flux of heat across boundaries $+$ Total heat generated inside [x, x+Δx]



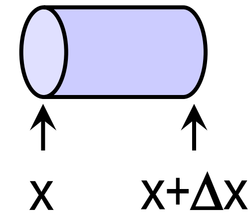
- Net change of heat is equal to:

$$\frac{d}{dt}\left[\int_{x}^{x+\Delta x} \rho \cdot C_p \cdot A \cdot T(s,t) \cdot ds\right] = \rho \cdot C_p \cdot A \cdot \Delta x \cdot \frac{\partial T(x,t)}{\partial t}$$

(Assume $\Delta x \to 0$)

Conservation of Heat

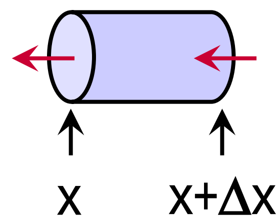
Net change of heat inside [x, x+Δx] = Net flux of heat across boundaries + Total heat generated inside [x, x+Δx]



- Fourier's law: heat flux across a boundary is proportional to the temperature gradient across the boundary:

$$\kappa \cdot A \cdot \frac{\partial T(x,t)}{\partial x}$$

Heat flux at x



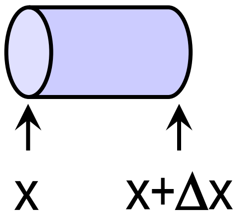
$$\kappa \cdot A \cdot \frac{\partial T(x+\Delta x,t)}{\partial x}$$

Heat flux at x+Δx

κ:	thermal conductivity (measure the ability to conduct heat)
A:	cross-section area
T:	temperature
t:	time
x:	x-coordinate

Conservation of Heat

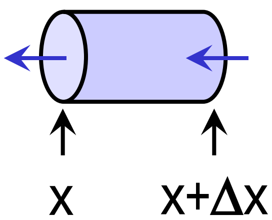
Net change of heat inside [x, x+Δx] = Net flux of heat across boundaries + Total heat generated inside [x, x+Δx]



- Net flux of heat across boundaries is equal to:

$$\kappa \cdot A \cdot \frac{\partial T(x,t)}{\partial x}$$

Heat flux at x



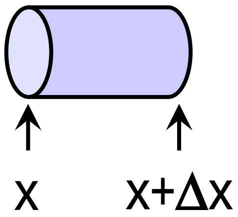
$$\kappa \cdot A \cdot \frac{\partial T(x+\Delta x,t)}{\partial x}$$

Heat flux at x+Δx

$$\kappa \cdot A \cdot \left[\frac{\partial T(x+\Delta x,t)}{\partial x} - \frac{\partial T(x,t)}{\partial x} \right]$$

Conservation of Heat

Net change of heat inside [x, x+Δx] = Net flux of heat across boundaries + Total heat generated inside [x, x+Δx]



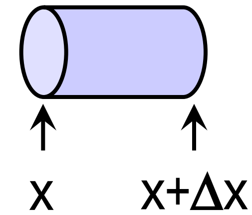
- Total heat generated inside [x, x+Δx] is equal to:

$$\int_{x}^{x+\Delta x} A \cdot f\left(s,t\right) \cdot ds$$

A:	cross-section area
f:	heat source
t:	time
s:	x-coordinate

1-D Heat Equation

Net change of heat inside [x, x+Δx] = Net flux of heat across boundaries + Total heat generated inside [x, x+Δx]



- Overall, we have:

$$\rho \cdot C_p \cdot A \cdot \Delta x \cdot \frac{\partial T(x,t)}{\partial t} = \kappa \cdot A \cdot \left[\frac{\partial T(x+\Delta x,t)}{\partial x} - \frac{\partial T(x,t)}{\partial x} \right] + \int_x^{x+\Delta x} A \cdot f(s,t) \cdot ds$$

⇓

$$\rho \cdot C_p \cdot \frac{\partial T(x,t)}{\partial t} = \kappa \cdot \frac{1}{\Delta x} \left[\frac{\partial T(x+\Delta x,t)}{\partial x} - \frac{\partial T(x,t)}{\partial x} \right] + \frac{1}{\Delta x} \cdot \int_x^{x+\Delta x} f(s,t) \cdot ds$$

1-D Heat Equation

$$\rho \cdot C_p \cdot \frac{\partial T(x,t)}{\partial t} = \kappa \cdot \frac{1}{\Delta x}\left[\frac{\partial T(x+\Delta x,t)}{\partial x} - \frac{\partial T(x,t)}{\partial x}\right] + \frac{1}{\Delta x} \cdot \int_x^{x+\Delta x} f(s,t) \cdot ds$$

$\Delta x \rightarrow 0$

$$\rho \cdot C_p \cdot \frac{\partial T(x,t)}{\partial t} = \kappa \cdot \frac{\partial^2 T(x,t)}{\partial x^2} + f(x,t)$$

$$\rho \cdot C_p \cdot T_t(x,t) = \kappa \cdot T_{xx}(x,t) + f(x,t)$$

Partial differential equation (PDE)

Partial Differential Equation (PDE)

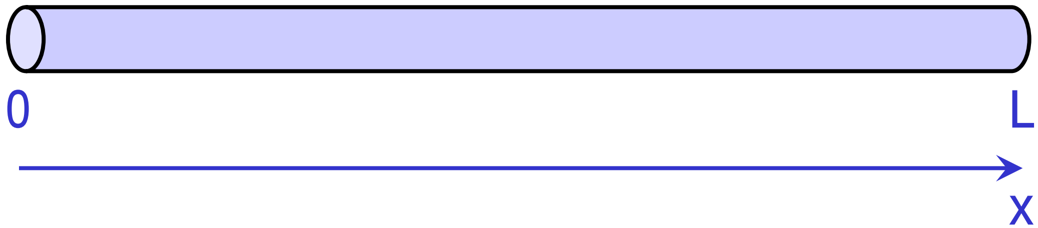
$$\rho \cdot C_p \cdot T_t(x,t) = \kappa \cdot T_{xx}(x,t) + f(x,t)$$

- T (which we differentiate) is called the dependent variable
- t and x (which we differentiate with respect to) are called independent variables
- In addition to this PDE, we further need to know the boundary and initial conditions to uniquely determine T(x,t)

Boundary Conditions

- Boundary conditions describe the physical nature of our problem on the boundaries

- A simple example
  - Rod temperature is fixed at the two ends

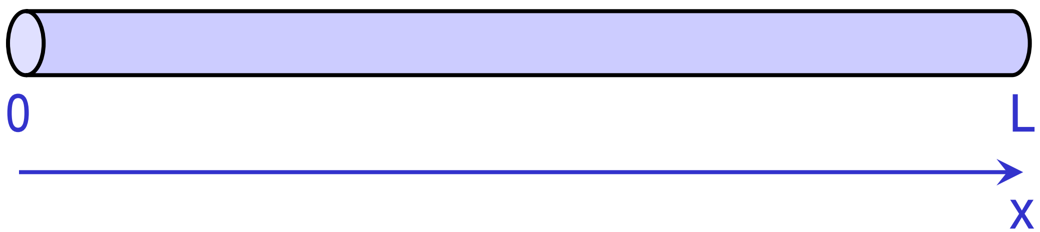


$$T(x=0,t)=T_1$$
$$T(x=L,t)=T_2$$

Initial Conditions

- Initial conditions describe the physical phenomenon at the beginning of the thermal transfer process

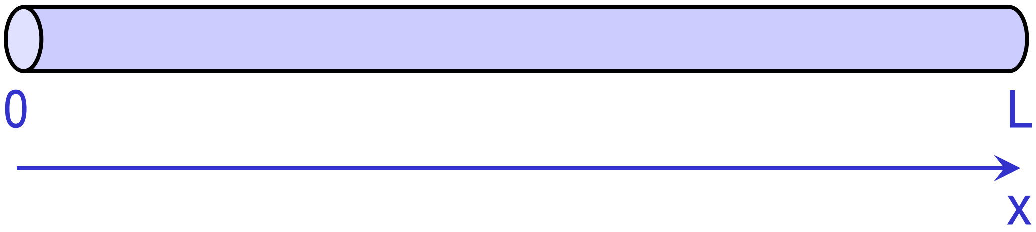
- A simple example
  - Rod is initially at an equilibrium point – constant temperature



$$T\left(x, t=0\right) = T_0$$

1-D Heat Equation

- The complete PDE with boundary and initial conditions



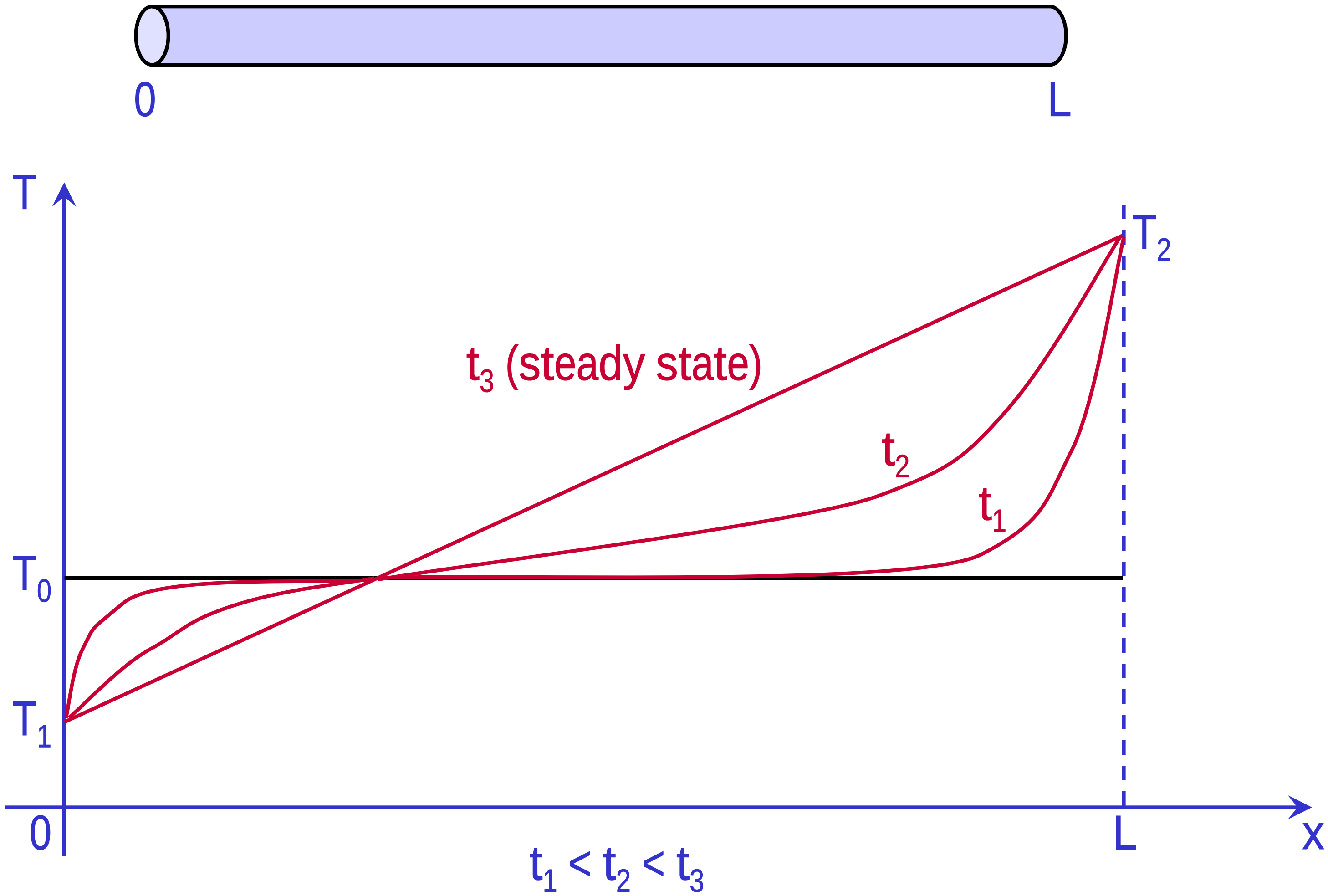
PDE $$\rho \cdot C_p \cdot T_t(x,t) = \kappa \cdot T_{xx}(x,t) + f(x,t) \quad (0 \le x \le L \quad 0 \le t \le \infty)$$

BCs $$\begin{cases} T(x=0,t) = T_1 \\ T(x=L,t) = T_2 \end{cases} \quad (0 < t \le \infty)$$

IC $$T(x,t=0) = T_0 \quad (0 \le x \le L)$$

1-D Heat Equation

- The solution of this PDE can be analytically calculated



Boundary Conditions

- There are many ways to specify boundary conditions
- Option 1: temperature is specified on boundaries

$$T(x=0,t)=g_1(t)$$
$$T(x=L,t)=g_2(t)$$

Boundary Conditions

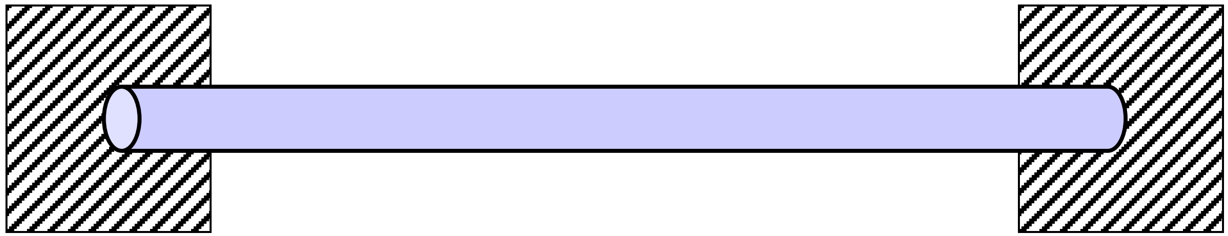
- The BCs used for our 1-D rod belongs to this category

$$T(x=0,t)=T_1$$
$$T(x=L,t)=T_2$$

- Practically useful to force the temperature to behave in a suitable manner
  - E.g., boundary control in steel industry

Boundary Conditions

- Option 2: temperature of the surrounding medium is specified



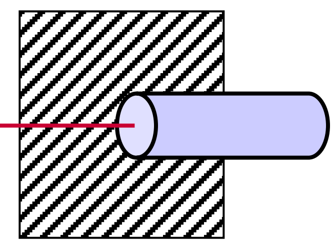
Liquid kept at temperature $g_1(t)$

Liquid kept at temperature $g_2(t)$

  - Newton's law: heat flux across the boundary is proportional to the temperature difference

$$h \cdot A \cdot \left[T\left(x=0,t\right) - g_1\left(t\right)\right]$$

Heat flux at x = 0

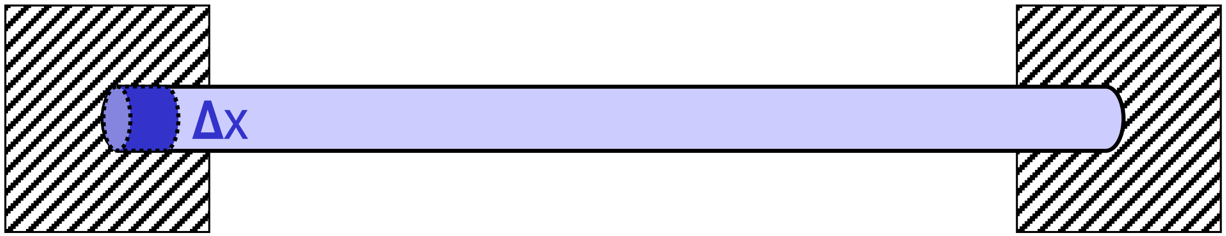


Liquid kept at temperature $g_1(t)$

h: heat-exchange coefficient (measure how fast heat flows across boundary)

Boundary Conditions

- Consider Newton's and Fourier's laws at x = 0

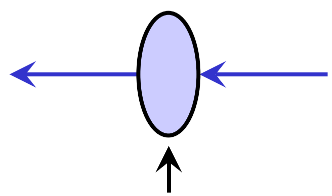


Liquid kept at temperature $g_1(t)$

Liquid kept at temperature $g_2(t)$

$$h \cdot A \cdot \left[T(x=0,t) - g_1(t)\right]$$

Heat flux at x = 0
(Newton's law)



x = 0

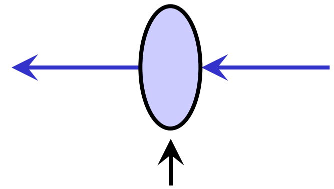
$$\kappa \cdot A \cdot \left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0}$$

Heat flux at x = 0
(Fourier's law)

Boundary Conditions

$$h \cdot A \cdot \left[T\left(x=0,t\right) - g_1\left(t\right)\right]$$

Heat flux at x = 0
(Newton's law)



x = 0

$$\kappa \cdot A \cdot \left.\frac{\partial T\left(x,t\right)}{\partial x}\right|_{x=0}$$

Heat flux at x = 0
(Fourier's law)

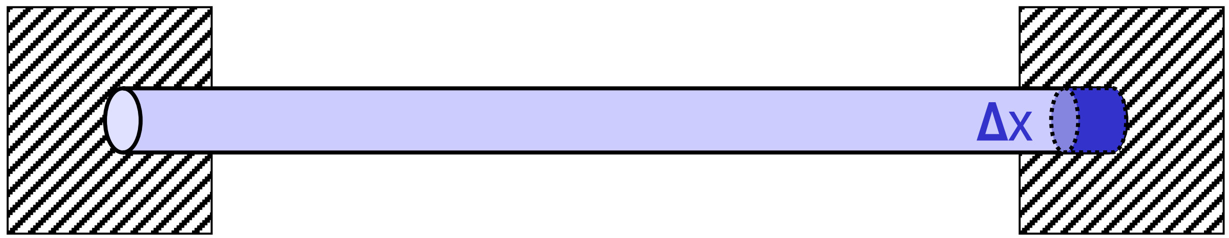
- Apply conservation of heat flux to the boundary x = 0

$$\kappa \cdot A \cdot \left.\frac{\partial T\left(x,t\right)}{\partial x}\right|_{x=0} = h \cdot A \cdot \left[T\left(x=0,t\right) - g_1\left(t\right)\right]$$

$$\kappa \cdot \left.\frac{\partial T\left(x,t\right)}{\partial x}\right|_{x=0} = h \cdot \left[T\left(x=0,t\right) - g_1\left(t\right)\right]$$

Boundary Conditions

- Similarly, we can derive the boundary condition at x = L

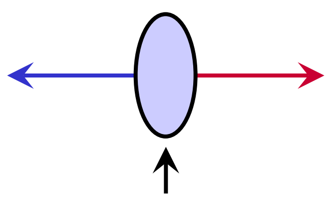


Liquid kept at temperature $g_1(t)$

Liquid kept at temperature $g_2(t)$

$$\kappa \cdot A \cdot \left.\frac{\partial T(x,t)}{\partial x}\right|_{x=L}$$

Heat flux at x = L
(Fourier's law)



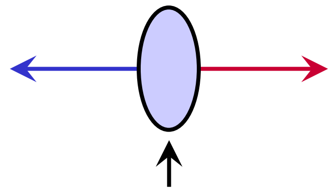
$$h \cdot A \cdot \left[T(x=L,t) - g_2(t)\right]$$

Heat flux at x = L
(Newton's law)

Boundary Conditions

$$\kappa \cdot A \cdot \left.\frac{\partial T(x,t)}{\partial x}\right|_{x=L}$$

Heat flux at x = L
(Fourier's law)



x = L

$$h \cdot A \cdot \left[T(x=L,t) - g_2(t)\right]$$

Heat flux at x = L
(Newton's law)

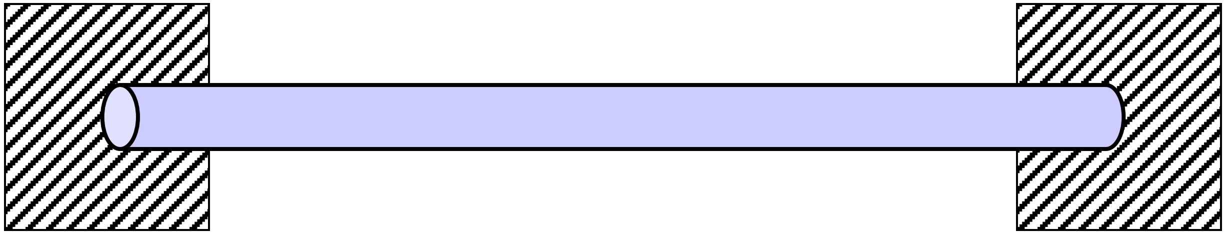
- Apply conservation of heat flux to the boundary x = L

$$\kappa \cdot A \cdot \left.\frac{\partial T(x,t)}{\partial x}\right|_{x=L} = -h \cdot A \cdot \left[T(x=L,t) - g_2(t)\right]$$

$$\kappa \cdot \left.\frac{\partial T(x,t)}{\partial x}\right|_{x=L} = -h \cdot \left[T(x=L,t) - g_2(t)\right]$$

Boundary Conditions

- Option 2: temperature of the surrounding medium is specified



Liquid kept at temperature $g_1(t)$

Liquid kept at temperature $g_2(t)$

$$\kappa \cdot \left. \frac{\partial T(x,t)}{\partial x} \right|_{x=0} = h \cdot \left[T(x=0,t) - g_1(t) \right]$$

$$\kappa \cdot \left. \frac{\partial T(x,t)}{\partial x} \right|_{x=L} = -h \cdot \left[T(x=L,t) - g_2(t) \right]$$

Boundary Conditions

- Option 3: heat flow across the boundaries is specified

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = g_1(t)$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=L} = g_2(t)$$

Boundary Conditions

- Insulated boundaries (also referred to as reflective boundaries)
  - No heat passes through boundaries

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=0} = 0$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=L} = 0$$

Summary

- Partial differential equation (PDE)
  - Heat equation
  - Boundary condition