18-660: Numerical Methods for Engineering Design and Optimization

Xin Li
Department of ECE
Carnegie Mellon University
Pittsburgh, PA 15213

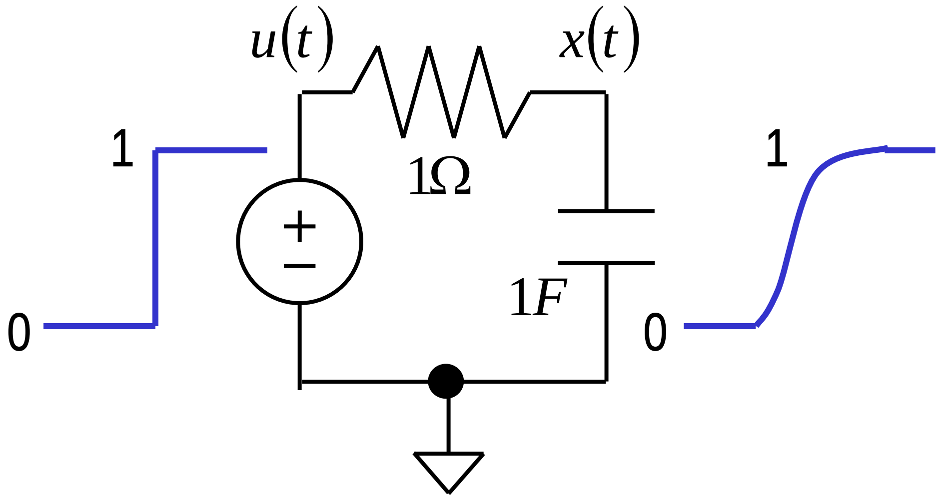
Electrical & Computer ENGINEERING

Overview

- Ordinary Differential Equation (ODE)
  - Numerical integration
  - Stability

Ordinary Differential Equation (ODE)

- Transient analysis for electrical circuit



$$\dot{x}(t) = u(t) - x(t)$$ → Ordinary differential equation

$$x(0) = 0$$ → Initial condition

$$u(t) = 1 \quad (t \geq 0)$$

Ordinary Differential Equation (ODE)

- General mathematical form

$$F\left[\dot{x}(t), x(t), u(t)\right] = 0 \quad x(0) = X$$

$x(t)$: N-dimensional vector of unknown variables

$u(t)$: Vector of input sources

F : Nonlinear operator

X : Initial condition

Numerical Integration

- In general, closed-form solution does not exist
  - Even if ODE is linear, we cannot find analytical solutions in many practical cases

- Numerical methods must be applied to approximate the solution – numerical solution
  - Numerical integration for differential operator

$$\dot{x}(t) \approx ???$$

↓

Algebraic equation

Numerical Integration

- Several different formulas exist for numerical integration
  - One-step numerical integration approximates differential operator from two successive time points

$$\dot{x} = f(x)$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f(x)$$

Forward Euler (FE)

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f\left[x(t_n)\right] = f(t_n)$$

Backward Euler (BE)

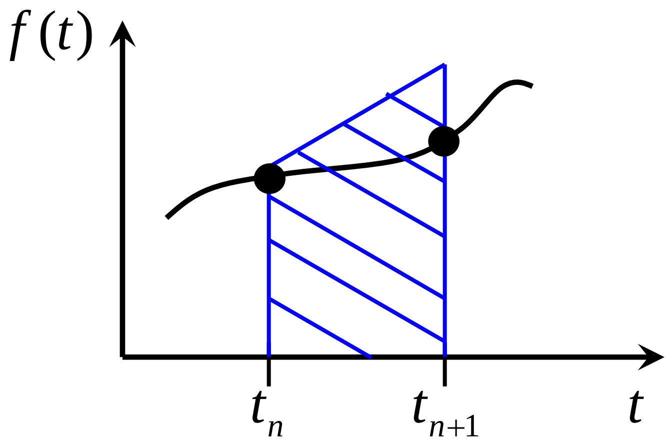
$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f(t_{n+1})$$

Trapezoidal (TR)

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx \frac{f(t_n) + f(t_{n+1})}{2}$$

Trapezoidal Approximation

- Trapezoidal is often more accurate but also more expensive than BE and FE



$$\dot{x} = f(x)$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx \frac{f[x(t_n)] + f[x(t_{n+1})]}{2}$$

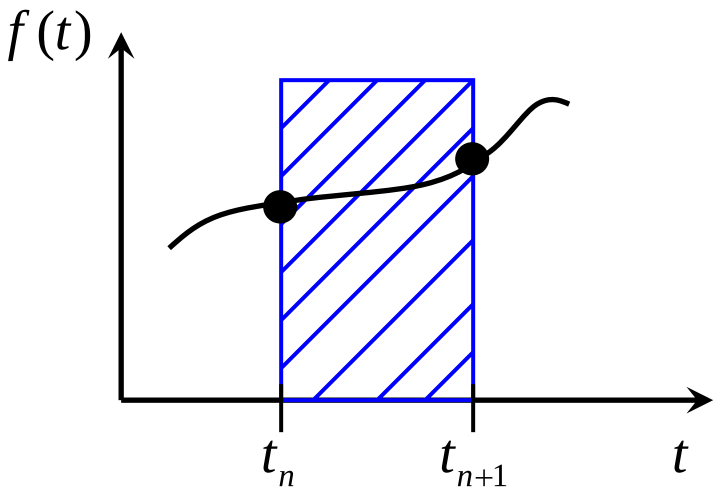
$$x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f[x(t)] \cdot dt$$

$$\approx \Delta t \cdot \frac{f[x(t_n)] + f[x(t_{n+1})]}{2}$$

Backward Euler Approximation

- Similar to TR but is less accurate and expensive
- Widely used for practical applications

$$\dot{x} = f(x)$$



$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f\left[x(t_{n+1})\right]$$

$$x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f\left[x(t)\right] \cdot dt$$

$$\approx \Delta t \cdot f\left[x(t_{n+1})\right]$$

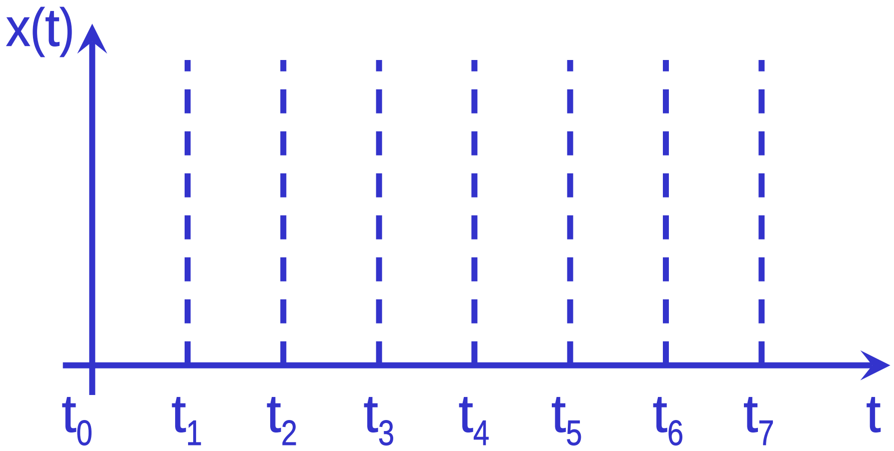
Backward Euler Example

- First-order system example

$$\dot{x}(t) = u(t) - x(t)$$
$$x(0) = 0$$
$$u(t) = 1 \quad (t \geq 0)$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = 1 - x(t_{n+1})$$
$$x(t_0) = 0$$

$$\dot{x} = f(x)$$
$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f(t_{n+1})$$



Backward Euler Example

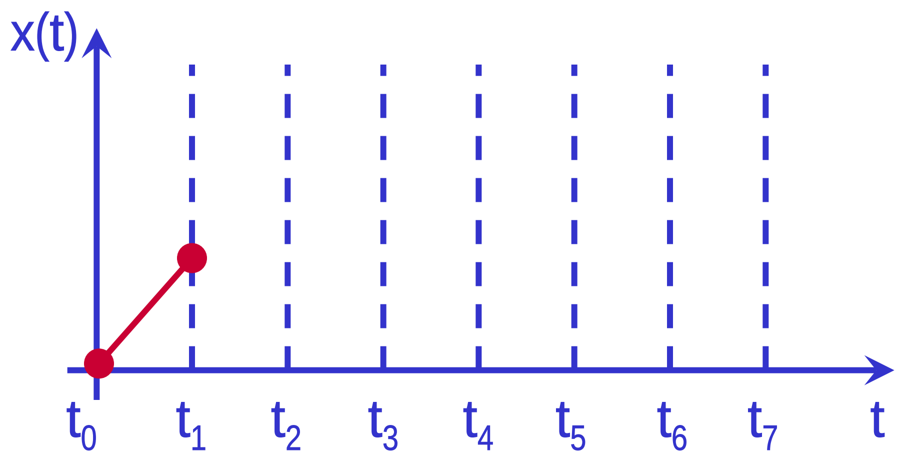
- First-order system example

$$\frac{x(t_1) - x(t_0)}{\Delta t} = 1 - x(t_1)$$
$$x(t_0) = 0$$

$$x(t_1) - x(t_0) = \Delta t - \Delta t \cdot x(t_1)$$

$$x(t_1) = \frac{\Delta t + x(t_0)}{1 + \Delta t} = \frac{\Delta t}{1 + \Delta t}$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = 1 - x(t_{n+1})$$
$$x(t_0) = 0$$



Backward Euler Example

- First-order system example

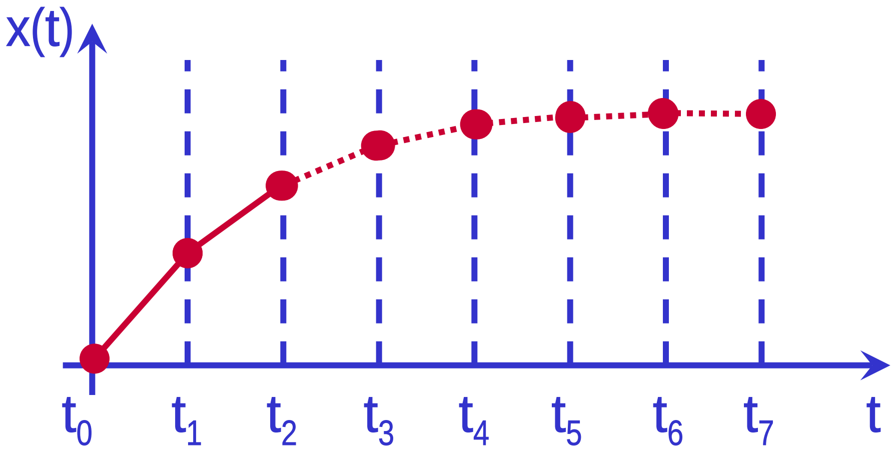
$$\frac{x(t_2)-x(t_1)}{\Delta t}=1-x(t_2)$$

$$x(t_2)-x(t_1)=\Delta t-\Delta t\cdot x(t_2)$$

$$x(t_2)=\frac{\Delta t+x(t_1)}{1+\Delta t}$$

$$\frac{x(t_{n+1})-x(t_n)}{\Delta t}=1-x(t_{n+1})$$

$$x(t_0)=0$$

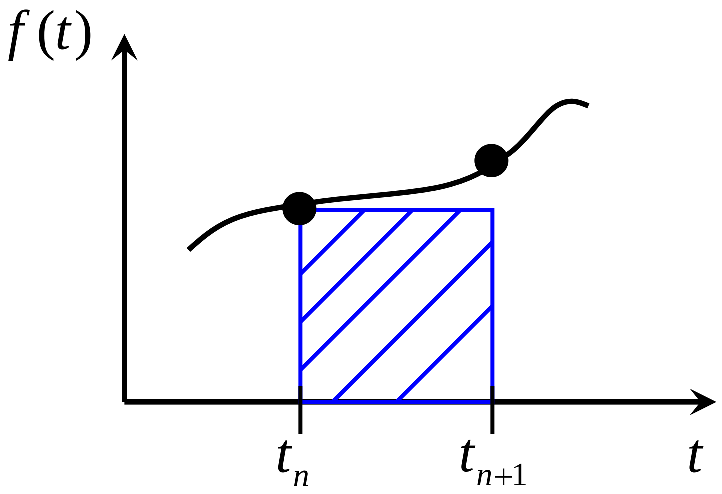


Continue iteration to determine x(t)

Forward Euler Approximation

- Least accurate compared to TR and BE
- Difficult to guarantee stability

$$\dot{x} = f(x)$$



$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f[x(t_n)]$$

$$x(t_{n+1}) - x(t_n) = \int_{t_n}^{t_{n+1}} f[x(t)] \cdot dt$$

$$\approx \Delta t \cdot f[x(t_n)]$$

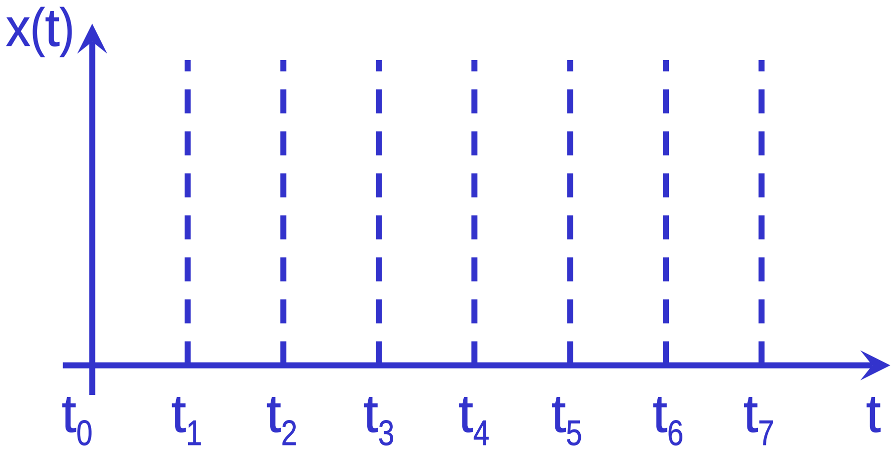
Forward Euler Example

- First-order system example

$$\dot{x}(t) = u(t) - x(t)$$
$$x(0) = 1 \quad u(t) = 0$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = -x(t_n)$$
$$x(t_0) = 1$$

$$\dot{x} = f(x)$$
$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} \approx f(t_n)$$



Forward Euler Example

- First-order system example

$$\frac{x(t_1)-x(t_0)}{\Delta t}=-x(t_0)$$

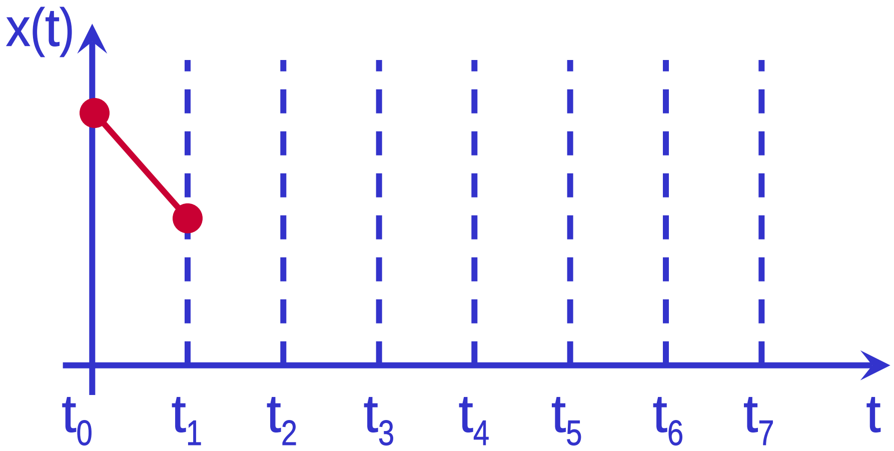
$$x(t_0)=1$$

$$x(t_1)=-\Delta t+1$$

$$x(t_1)=1-\Delta t$$

$$\frac{x(t_{n+1})-x(t_n)}{\Delta t}=-x(t_n)$$

$$x(t_0)=1$$



Forward Euler Example

- First-order system example

$$\frac{x(t_2) - x(t_1)}{\Delta t} = -x(t_1)$$

$$x(t_1) = 1 - \Delta t$$

$$\Downarrow$$

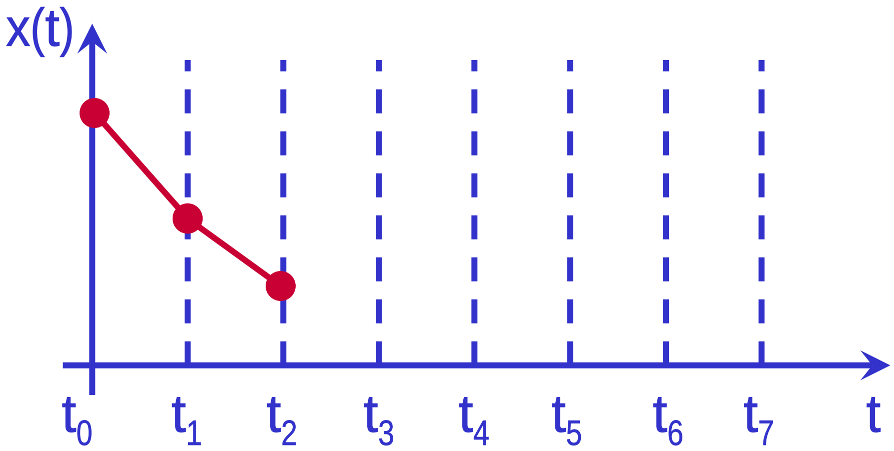
$$x(t_2) = -\Delta t \cdot x(t_1) + x(t_1)$$

$$\Downarrow$$

$$x(t_2) = (1 - \Delta t) \cdot x(t_1)$$
$$= (1 - \Delta t)^2$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = -x(t_n)$$

$$x(t_0) = 1$$



Forward Euler Example

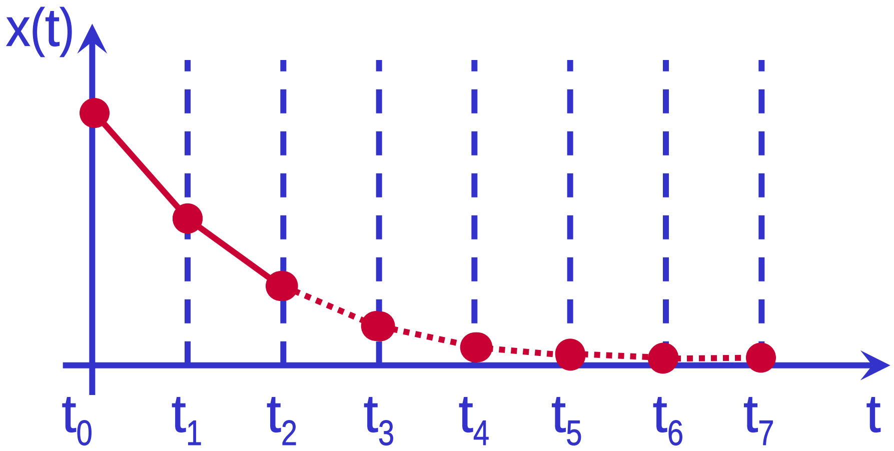
- First-order system example

$$\frac{x(t_{n+1})-x(t_n)}{\Delta t} = -x(t_n)$$
$$x(t_0)=1$$

$$x(t_3)=(1-\Delta t)^3$$

$$x(t_4)=(1-\Delta t)^4$$

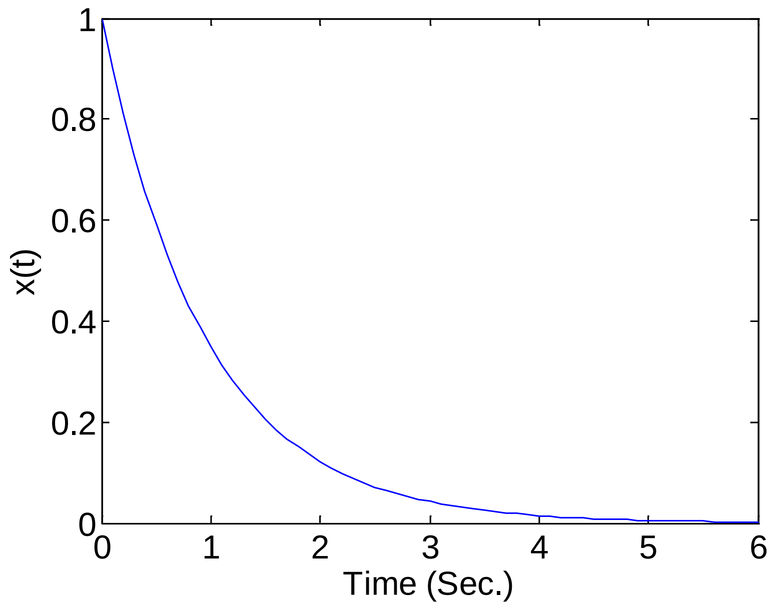
$$x(t_n)=(1-\Delta t)^n$$



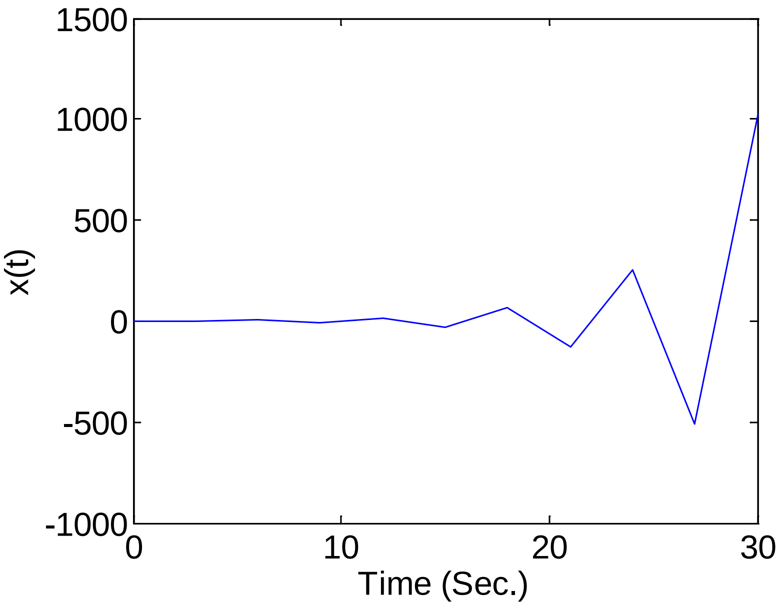
Forward Euler Example

- First-order system example

$$x(t_n) = (1 - \Delta t)^n$$



Δt = 0.1 (correct answer)



Δt = 3 (fail to converge)

Forward Euler fails to converge if Δt is too large

Numerical Integration Stability

- Δt must be sufficiently small for FE to guarantee stability
  - In practice, it is not easy to determine the appropriate Δt
- BE and TR do not suffer from stability issue
  - Stability is guaranteed for any $\Delta t > 0$
- First-order system example

$$\dot{x}(t) = u(t) - x(t)$$
$$x(0) = 1 \quad u(t) = 0$$

BE →

$$x(t_n) = \frac{1}{(1+\Delta t)^n}$$

Always stable for $\Delta t > 0$

Nth-Order Linear ODE

- Our simple example solves a first-order linear ODE

$$\dot{x}(t) = u(t) - x(t)$$

- In general, an Nth-order linear time-invariant dynamic system is described by the following ODE:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$ → Ordinary differential equation

$$x(0) = 0$$ → Initial condition

$x(t)$: N-dimensional vector of unknown variables

$u(t)$: Vector of input sources

A, B: Matrices

Nth-Order Linear ODE

- Backward Euler example

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t) \quad x(0) = 0$$

$$\Downarrow$$

$$\frac{x(t_{n+1}) - x(t_n)}{\Delta t} = A \cdot x(t_{n+1}) + B \cdot u(t_{n+1}) \quad x(t_0) = 0$$

$$\Downarrow$$

$$x(t_{n+1}) - x(t_n) = \Delta t \cdot A \cdot x(t_{n+1}) + \Delta t \cdot B \cdot u(t_{n+1}) \quad x(t_0) = 0$$

$$\Downarrow$$

$$x(t_{n+1}) = (I - \Delta t \cdot A)^{-1} \cdot [x(t_n) + \Delta t \cdot B \cdot u(t_{n+1})] \quad x(t_0) = 0$$

Solve linear algebraic equation to find $x(t_{n+1})$

Nth-Order Nonlinear ODE

- Many physical systems are both high-order and nonlinear

$$F\left[\dot{x}(t), x(t), u(t)\right] = 0 \quad x(0) = 0$$

$x(t)$: N-dimensional vector of unknown variables

$u(t)$: Vector of input sources

F : Nonlinear operator

Nth-Order Nonlinear ODE

- Backward Euler example

$$F\left[\dot{x}(t), x(t), u(t)\right] = 0 \quad x(0) = 0$$

$$\Downarrow$$

$$F\left[\frac{x(t_{n+1}) - x(t_n)}{\Delta t}, x(t_{n+1}), u(t_{n+1})\right] = 0 \quad x(t_0) = 0$$

Solve nonlinear algebraic equation to find $x(t_{n+1})$

- Solving nonlinear algebraic equation requires iterative algorithm
  - More details in future lectures...

Advanced Topics for ODE Solver

- Local truncation error estimation
  - Estimate approximation error for numerical integration
- Adaptive time step control
  - Dynamically determine Δt
- High-order integration formula
  - Apply multi-step numerical integration
- Some of these advanced topics are covered by 18-762 that particularly focuses on ODE solver for circuit simulation

Summary

- Ordinary differential equation (ODE)
  - Numerical integration
  - Stability