Consensus Based Detection in Sensor Networks: Topology Optimization under Practical Constraints

Soummya Kar[1] and José M. F. Moura[2]*

[1] Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA, 15213, USA,
soummyak@andrew.cmu.edu

[2] Department of ECE, Carnegie Mellon University, Pittsburgh, PA, 15213, USA,
moura@ece.cmu.edu

Abstract. We consider a distributed hypothesis testing problem in sensor networks with possibly correlated sensor observations. The intersensor communication is constrained by the underlying communication network to sensors exchanging information only with their neighbors. The network has no central fusion center. We show that, under reasonable assumptions, the global test statistic can be computed locally at each sensor by a distributed iterative consensus algorithm. We consider the problem of optimizing the sensor network topology with respect to the rate of convergence of iterative consensus under different practical design constraints. For nonrandom topologies with fixed intersensor communication costs, the class of Ramanujan graphs is optimal when the underlying communication graph is regular, the communication is noiseless, and the intersensor communication costs are constant across the network. In contrast, when communication among links exhibits different costs, links may fail, the optimal topology under an overall communication cost constraint is obtained by solving a semidefinite programming optimization problem.

Key words: Sensor Networks, Distributed Detection, Spectral Graph Theory, Graph Laplacian, Algebraic Connectivity, Topology Optimization, Ramanujan Graphs, Communication Constraints.

1 Introduction

We consider a problem of distributed detection with a network of N sensors. Each sensor makes an observation and computes its local decision statistic. To reach a more reliable global decision, the network fuses the individual local statistics. Traditional network architectures consider a central fusion center: the individual

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sensors send their local observations (quantized or unquantized) to the fusion center that then computes the global decision. This communication architecture requires only $O(N)$ communication for every sensor to learn the global decision: in step 1, the sensors transmit their observation to the center node, and in step 2, the center broadcasts to all sensors its decision. But this architecture is not practical in many scenarios. For example, for N large, the fusion center is an information bottleneck that may cause the system to collapse. In this paper, we restrict the communication among sensors by a connectivity graph where each sensor can exchange information only with its neighbors. Under reasonable assumptions on the network connectivity graph, each sensor can iteratively achieve the performance of the optimal global test (with a central fusion center) in a distributed manner by a consensus averaging algorithm.

Distributed consensus is a well-studied problem with an extensive literature (see, for example [1–3] and the references therein.) This literature does not address the problem of optimizing the sensor network topology, under different constraints. The paper describes our studies on how the underlying sensor network topology impacts the rate of convergence of this iterative distributed decision, illustrating how some networks converge orders of magnitude faster than others.

Section 4 considers distributed detection in a fixed (non-time varying) topology environment and designs the optimum topology, subject to a constraint on the number of network links (or equivalently a constraint on the total communication cost, assuming equal costs across different network links.) Prior work on this (see [4, 5]) restricted attention to classes of random networks, particularly, small-world networks. In [6, 7], we show that the class of non-bipartite Ramanujan graphs are essentially optimal for this scenario and perform much better than graphs with nearest neighbor topology, as well as other random networks (see also [8].) In Section 5, we extend the framework of Section 4 to analyze the detection problem in sensor networks with noisy communication channels. In particular, we assume that at each iteration, the data exchange is corrupted by additive noise. We study this problem in [9] where we showed that there exists an optimal number of iterations before reaching a decision. Section 6 considers another very practical communication channel model in sensor networks where the quality of a communication channel between two sensors is quantified by the probability of reliable transmission, which in turn is determined by the signal-to-noise ratio (SNR) allocated to the channel. In a typical scenario, a network transmission is either received correctly or the channel decides an error. It is also reasonable to assume that the errors across different channels occur independently of each other and across time. In this Section, we model such unreliable communication by assigning to each network link a probability of correct reception. This results in a network model where the links fail randomly independent of each other and across time. We analyze the convergence properties of the consensus algorithm in this scenario (see [10, 11]) subject to a constraint on the total power or communication cost in the network. We describe how this problem can be formulated as a convex optimization problem, thus admitting efficient numerical solutions.

2 Background

The sensor network topology is captured by an undirected graph $G = (V, E)$. The set V collects the sensor nodes with $|V| = N$. The set E is the set of graph edges or links with edge $(n, l) \in E$ *iff* there is a link or communication channel between nodes n and l. We let $|E| = M$. For any graph G, we can define the following standard objects in graph theory. The $N \times N$ adjacency matrix A is

$$A_{n,l} = \begin{cases} 1 \text{ if } (n,l) \in E \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

The neighborhood of a vertex or node is

$$\Omega_n = \{l \in V : \ (n,l) \in E\}, \ \forall n \in [1, ..., N] \tag{2}$$

The degree of a node is the number of edges incident to it

$$d_n = |\Omega_n|, \ \forall n \in [1, ..., N] \tag{3}$$

The $N \times N$ Laplacian matrix L is

$$L = D - A \tag{4}$$

where $D = diag(d_1, ..., d_N)$ is the degree matrix. The Laplacian L is a symmetric positive-semidefinite matrix. We order its eigenvalues as,

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq ... \leq \lambda_N(L) \tag{5}$$

For a connected graph $\lambda_2(L) > 0$ (see [12].) A graph where all nodes have the same degree d is d-regular.

3 Distributed Detection

We formulate the distributed detection problem. Consider the binary hypothesis testing problem

$$\text{under } \mathcal{H}_p : \ \mathbf{y} = \mathbf{m}_p + \xi, \ p = 0, 1 \tag{6}$$

where $\mathbf{y} \in \mathbb{R}^{N \times 1}$ denotes the vector of sensor observations. We assume $\mathbf{m}_1 = -\mathbf{m}_0 = \mathbf{m}$ and

$$\xi \ \sim \ \mathcal{N}(\mathbf{0}, K) \tag{7}$$

where, $\mathbf{0} \in \mathbb{R}^{N \times 1}$ is the vector of all zeros and K is the positive definite covariance matrix of ξ.

In a centralized fusion architecture, the minimum probability of error Bayes detector is

$$l(\mathbf{y}) = \frac{2}{N}\mathbf{m}^T K^{-1} \mathbf{y} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} 0 \tag{8}$$

where $l(\mathbf{y})$ is the (global) sufficient statistic. Thus implementing the optimum global test is equivalent to computing the statistic $l(\mathbf{y})$.

We now discuss the distributed version of this centralized detector by noting that eqn. (8) can be realized at each individual sensor through distributed averaging. Rewrite $l(\mathbf{y})$ in (8) as

$$\frac{1}{N}\sum_{n=1}^{N} l_n(y_n) \tag{9}$$

where $l_n(y_n) = \omega_n y_n,\ i = 1, ..., N$, are local statistics, i.e., can be computed at each sensor from their own measurement. Distributed detection is now equivalent to computing in a distributed fashion the average (9). This can be done by average consensus algorithms.

Although a simplified observation model is considered here, it should be noted that consensus based detection is applicable to a much wider class of models, having a global sufficient statistic that is a separable function of the observations (see also [**?**] and the references therein for general separable function computation.)

4 Fixed Sensor Network Topology with Noiseless Communication Channels

In this section, we consider the distributed detection problem in sensor networks with fixed (not varying with time) topology. Section 3 reduces the detection problem to an average consensus problem. The performance of the distributed detector is then determined by the convergence rate of the equivalent average consensus algorithm, which in turn is a function of the underlying sensor network topology, as we have considered in our previous work [6, 7, 9, 11]. Subsection 4.1 reviews the distributed average consensus algorithm and analyzes the influence of network topology on the convergence rate of the consensus algorithm. Subsection 4.2 presents relevant performance bounds for the distributed detection (and hence consensus) problem and motivates the topology optimization problem, the performance metric being the rate of convergence. In Subsection 4.3, we describes our work, [6, 7, 9], that shows that the class of non-bipartite Ramanujan graphs are essentially optimal when we constrain the number of network links (equivalently, constrain the total communication cost, assuming equal costs across all links.) Finally, Subsection 4.4 addresses the construction of Ramanujan graphs.

4.1 Distributed Average Consensus

In its simplest form, the distributed average consensus algorithm involves distributed linear iterations to compute the global average of local data. For our case, $\mathbf{x}(0) = [x_1(0) \cdots x_N(0)]$ is the vector of initial sensor observations. Denote the vector of averages as

$$\mathbf{x}_{\text{avg}} = \overline{r}\mathbf{1} \tag{10}$$

where $\mathbf{1}$ is the vector of ones and $\overline{r}$ is the average of initial data given by

$$\overline{r} = \frac{1}{N}\mathbf{1}^T\mathbf{x}(0) \tag{11}$$

The goal is to compute the average $\overline{r}$ at each sensor using distributed linear iterations involving only local communications. To this end, the sensors perform local state updates by

$$x_n(i+1) = W_{nn}x_n(i) + \sum_{l\in\Omega_n} W_{nl}x_l(i),\ n = \{1, ..., N\} \tag{12}$$

or equivalently in vector-matrix format as

$$\mathbf{x}(i+1) = W\mathbf{x}(i) \tag{13}$$

where W is a weight matrix. The non-zero entries of W are determined by the underlying sensor network connectivity. In particular, if $n \neq l$, then $(n, l) \notin E$ implies $W_{nl} = 0$.

The algorithm converges, i.e.,

$$\lim_{i\to\infty} \|\mathbf{x}(i) - \mathbf{x}_{\text{avg}}\| = 0 \tag{14}$$

if

$$\lim_{i\to\infty} W^i = \frac{1}{N}J \tag{15}$$

where $J = (1/N)\mathbf{1}\mathbf{1}^T$. In [2], it is shown that, if the network is connected, there exists W satisfying eqn. (15). For a given network topology, the choice of the W matrix plays a significant role in determining the convergence rate of the consensus algorithm; reference [2] addresses the problem of finding the optimum W leading to the fastest convergence for a given network. A commonly used choice for link weights is the optimum equal weights; the corresponding W matrix is

$$W = I - \alpha L \tag{16}$$

with

$$\alpha = \frac{2}{\lambda_2(L) + \lambda_N(L)} \tag{17}$$

In this paper, we consider the optimum equal weights assignment. Then , we showed (see [7]) that the consensus algorithm progresses as

$$\|\mathbf{x}(i) - \mathbf{x}_{\text{avg}}\| \ \leq\ \rho^i \,\|\mathbf{x}(0) - \mathbf{x}_{\text{avg}}\| \tag{18}$$

where ρ is given by

$$\rho = \frac{1 - \lambda_2(L)/\lambda_N(L)}{1 + \lambda_2(L)/\lambda_N(L)} = \frac{1-\gamma}{1+\gamma} \tag{19}$$

It follows that for a connected graph $|\rho| < 1$ and the consensus algorithm converges in the sense given by eqn. (14). Also, eqn. 18 suggests that the smaller ρ is, the faster the convergence rate is. We note that, ρ being a function of only the network topology, it provides the influence of network topology on the convergence speed and we have

$$\text{fast convergence} \ \Rightarrow\ \text{small } \rho \ \Rightarrow\ \text{large } \gamma \tag{20}$$

4.2 Performance Bounds and Topology Optimization Criterion

It follows from eqn. (9) in Section 3 that, if we initialize the consensus algorithm by $x_n(0) = l_n(y_n),\ n \in [1 \cdots N]$, the sensor states converge to the global sufficient statistic $l(\mathbf{y})$, provided the network is connected. Consider the following local test at sensor n at time i:

$$x_n(i) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} 0 \tag{21}$$

Let $P_e^n(i)$ be the corresponding probability of error. Also let P_e be the probability of error associated with the optimal global test (eqn. 8). Then, assuming that the network is connected, the convergence of $\mathbf{x}(i)$ to $l(\mathbf{y})$ implies

$$\lim_{i\to\infty} P_e^n(i) = P_e,\ n = 1, ..., N \tag{22}$$

and the rate of convergence is determined by the convergence rate of the consensus algorithm, which in turn depends on the network topology through the factor γ, as given in eqn. (20). In fact, for the simplified case of $\mathbf{m} = \mu\mathbf{1}$ and $K = \sigma^2 I$, it follows from standard detection theory that

$$P_e = \text{erfc}\left(\frac{\mu\sqrt{N}}{\sigma}\right) \tag{23}$$

where $\text{erfc}(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-z^2/2}dz$. Also, it can be shown that (see [6, 9])

$$P_e \leq P_e^n(i) \leq \text{erfc}\left(\frac{\mu\sqrt{N}}{\sigma\sqrt{1+\rho^{2i}(N-1)}}\right) \tag{24}$$

which is just a formal restatement of eqn. (20).

Eq. (24) shows that as N tends to infinity the probability of error $P_e^n(i)$ of each local test does tend to the probability of error P_e of the global test, i.e., the test at the fusion center. This means that the problem that is left to consider is the rate at which

$$P_e^n(i) \longrightarrow P_e$$

i.e., what is the topology that leads to the fastest convergence rate for the consensus (and hence distributed detection) algorithm. This problem is only of interest if there is a constraint on the number of links, or else we obtain the trivial solution of the complete topology where every sensor is linked to every other sensor. To this end, define the average degree of a graph as

$$d_{\text{avg}} = \frac{2|E|}{N} \tag{25}$$

Also let M be the maximum permissible number of network links. The topology optimization problem is then stated as

$$\max\ \gamma = \frac{\lambda_2(L)}{\lambda_N(L)} \tag{26}$$

subject to the constraint

$$|E| \leq M \quad \text{or} \quad d_{\text{avg}} \leq \frac{2M}{N} \tag{27}$$

(clearly, $d_{\text{avg}} = d$ for d-regular graphs.) This is indeed a difficult combinatorial optimization problem and there is no closed form solution for finite N. However, using a spectral graph theory result by Alon and Boppana (see [13]) we have shown that the class of non-bipartite Ramanujan graphs are essentially asymptotically ($N \to \infty$) optimal; this is the content of the next subsection.

4.3 Ramanujan Graphs: Asymptotically Optimal

We describe here our results on the optimality of the class of non-bipartite Ramanujan graphs with respect to the convergence speed of the consensus (and hence distributed detection) algorithm. In other words, given a constraint on the average degree d_{avg}, the non-bipartite Ramanujan graphs maximize the eigenratio $\gamma = \lambda_2(L)/\lambda_N(L)$ as the number of sensors N goes to infinity. The rest of Subsection 4.3 gives a brief description of Ramanujan graphs and provides brief arguments about their optimality (for detailed analysis and discussion, the reader is referred to [9, 7].)

The Ramanujan graphs are a class of d-regular graphs with extreme expansion properties (see, for example, [14]). We define a family of d-regular graphs to be a sequence of d-regular graphs with the number of nodes $N \to \infty$. Then (see [9]) for each member (i.e., for every finite value of N) of a family of d-regular non-bipartite Ramanujan graphs

$$\gamma = \frac{\lambda_2(L)}{\lambda_N(L)} \geq \frac{d - 2\sqrt{d-1}}{d + 2\sqrt{d-1}} \tag{28}$$

However, it can be shown from a result by Alon and Boppana ([13]) that, the lower bound in eqn. (28) is in fact an asymptotic (as $N \to \infty$) upper bound for almost all classes of d-regular graphs (see [9].) This means that the class of non-bipartite d-regular Ramanujan graphs approach the bound from above, while other classes of d-regular graphs stay below the bound as $N \to \infty$. This establishes the asymptotic optimality of the non-bipartite Ramanujan graphs, with respect to the convergence rate, among the class of d-regular graphs (see [9] for full details.)

The last paragraph established the optimality of non-bipartite Ramanujan graphs among the class of regular graphs. We now present a brief argument that shows that heterogeneity in degree distribution does not favor a large value of γ, and hence the Ramanujan graphs are expected to perform better than most families of non-regular graphs also. Let $d_{\min}$ and $d_{\max}$ be the minimum and maximum node degrees, respectively. Then, it can be shown from results in spectral graph theory that (see [15])

$$\gamma = \frac{\lambda_2(L)}{\lambda_N(L)} \leq \frac{d_{\min}}{d_{\max}} \tag{29}$$

Thus, networks with large spread in the degree distribution (for example, scale free networks or small world networks) will not favor very high values of γ, and hence the search space for optimal topology may be restricted to regular graphs (see [6, 9] for discussions and numerical studies that corroborate this fact.)

We thus established that the class of non-bipartite Ramanujan graphs are essentially optimal (asymptotically) solutions of the topology optimization problem given in eqn. (26). In the following subsection we address the issue of explicit constructions of Ramanujan graphs.

4.4 Explicit Constructions of Ramanujan Graphs and Numerical Studies

Explicit constructions of d-regular Ramanujan graphs (both bipartite and non-bipartite) were given independently by Lubotzky-Phillips-Sarnak (LPS) [16] and Margulis [17], for the case $d-1$ is a prime. The LPS construction was extended to cover the cases where $d-1$ is a prime power by Morgenstern [18]. Recently, several probabilistic methods have been developed for constructing expander families. In particular, [19] develops a new graph product which constructs expanders of arbitrary degree and size with high probability.

Fig. 1 shows a 6-regular non-bipartite Ramanujan graph on $N = 42$ vertices, obtained by the LPS ([16]) construction.

In [7], we perform detailed numerical studies and the results show that, even for finite values of N, the non-bipartite Ramanujan graphs perform much better than graphs with nearest neighbor topology, or small world type graphs, or Erdös-Renýi random networks. Also, the relative performance of the Ramanujan graphs increases steadily as the number of sensors N increases, thus verifying the asymptotic arguments in Subsection 4.3.

5 Fixed Sensor Network Topology with Noisy Communication Channels

In this section, we extend the distributed detection framework of Section 4 to the case where the network links are noisy or non-ideal. That is, at each iteration, the individual sensor updates get corrupted by the channel noise. We model this imperfection in the data exchange step by the following noisy state update equation (see [9]):

$$\mathbf{x}(i+1) = W\mathbf{x}(i) + \mathbf{n}(i) \tag{30}$$

where $\{\mathbf{n}(i)\}_{i=0}^{\infty}$ is the i.i.d. Gaussian noise sequence with

$$\mathbf{n}(i) \ \sim \ \mathcal{N}(\mathbf{0}, R), \quad R = \text{diag}[\phi_1^2 \cdots \phi_N^2] \tag{31}$$

Such a model arises, for example, when the channel noises are independent of each other. Also, as in Section 4, we initialize the sensor states by $x_n(0) = l_n(y_n),\ n \in [1 \cdots N]$.

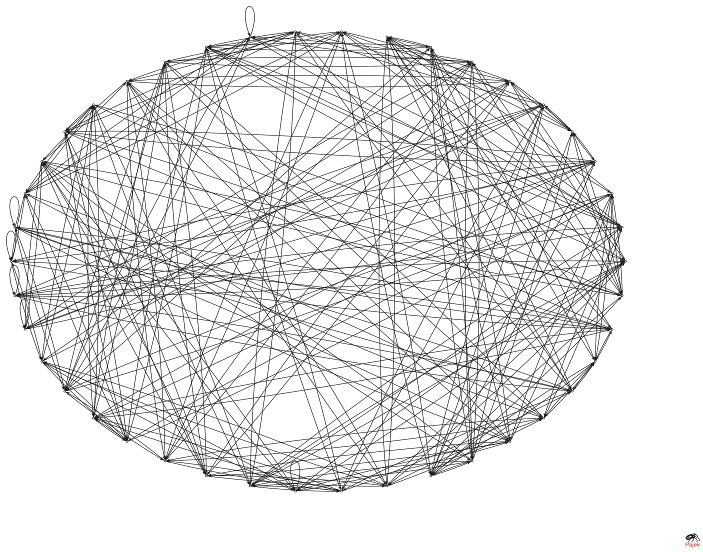

Fig. 1. Non-bipartite LPS Ramanujan graph with number of vertices $N = 42$ and degree $d = 6$ (figure generated using software Pajek.)

In [9], we analyze the above model. We present part of the results here without proof. It turns out, that, because of the additive noise incurred at each step, the sensor states do not converge to the global sufficient statistic required for the optimum global test. Thus, in practice, we can never achieve the performance of the optimal global detector, i.e., $P_e^n(i) \nrightarrow P_e$. But there exists an interesting trade-off between the number of iterations performed and the performance (in terms of local probability of error) achieved. Qualitatively, there exists an optimal number of iterations, i^*, after which the detrimental effect of noise addition overpowers the desired effect of mixing the local statistics. In particular, at each sensor n, $P_e^n(i)$ decreases till i^*, but starts increasing after that.

In [9] we actually compute this optimal number of iterations under a reasonable set of assumptions. The reader is also referred to [20] that studies the design of W for a given network topology for distributed consensus in additive noise.

6 Randomly varying Network Topology

In this section we consider the distributed consensus problem in sensor networks with randomly varying network topology (the reader may refer to the discussion in Section 1, which motivates this problem.) In Subsection 6.1, we formulate the problem and analyze the influence of network topology (statistics of link behavior) on the convergence properties of the consensus algorithm. Finally, Subsection 6.2 considers the problem of optimally allocating link failure probabilities (which may be viewed as proxies for the link SNRs), given a constraint on the total power or communication cost.

6.1 Consensus with Random Network Links: Influence of the Topology

Consider a network of N sensors. Let $\mathcal{E}$ denote the set of realizable edges, i.e., $\mathcal{E}$ defines the node pairs that can possibly exchange information. We model the unreliability of link (n, l) by assuming that it fails randomly with probability $1 - P_{nl}$. Collecting all these probabilities, we can define an $N \times N$ symmetric matrix, P, called the probability of edge formation matrix, given by

$$P_{nl} = \begin{cases} \text{Probability of edge } (n,l) \text{ if } (n,l) \in \mathcal{E} \\ 0 \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise} \end{cases} \tag{32}$$

This means that the edge set $E(i)$ at time i is a random subset of $\mathcal{E}$, chosen according to the P matrix. In other words, at each time i, the network graph is a random instantiation with edge set $E(i)$. Consequently, all the network matrices ($A(i)$,$L(i)$) are random matrices. It follows that the mean Laplacian, $\overline{L} = \mathrm{E}[L]$, is given by (see [10])

$$\overline{L}_{nl} = \begin{cases} \sum_{m=1}^{N} P_{nm} \text{ if } n = l \\ -P_{nl} \qquad\quad \text{otherwise} \end{cases} \tag{33}$$

In [10], we analyze the distributed consensus algorithm in such a network with the following state update:

$$\mathbf{x}(i + 1) = W(i)\mathbf{x}(i) \tag{34}$$

where

$$W(i) = I - \alpha L(i) \tag{35}$$

We note that the weight matrices are now random matrices because the networks $L(i)$ are random. The constant α is an user-defined parameter and must be chosen to guarantee satisfactory performance. Also, as usual, we initialize the sensor states by the local statistics $l_n(y_n)$. It follows from eqn. (34) that the sensor updates are now random and hence the convergence properties of the consensus algorithm are to be interpreted in some probabilistic sense.

We end this subsection by presenting a convergence result from [11] without proof.

Theorem 1. *A necessary and sufficient condition for mean square (mss) and almost sure (a.s.) convergence of the consensus algorithm is* $\lambda_2(\overline{L}) > 0$. *In other words, if* $\lambda_2(\overline{L}) > 0$*, then there exists an* α*, for which the consensus algorithm converges in mss or a.s sense.*

Thus Theorem 1 relates the convergence properties of the consensus algorithm in eqn. (34) to the statistics of the mean Laplacian matrix, $\overline{L}$, which in turn depends on the P matrix.

6.2 Topology Optimization under Communication Constraints

In Subsection 6.1 we analyzed the influence of the network link formation statistics on the convergence rate of the distributed consensus algorithm (see [10] that studies the influence of P on the mss convergence rate.) The discussion in Section 1 shows that the link formation probabilities may be viewed as proxies for the channel SNRs. Hence a relevant optimization problem is the optimal allocation of channel SNRs under a power or communication cost constraint. Since choosing the optimal SNRs is equivalent to designing the optimal P (or equivalently $\overline{L}$), we treat $\overline{L}$ as the optimization variable. Due to the lack of space, we do not consider this optimization problem in detail here, but refer the interested reader to [11]. In [11], we show that the optimization problem involves maximizing the algebraic connectivity of the mean Laplacian $\overline{L}$, under a set of convex constraints.

7 Conclusion

In this paper, we considered the distributed detection problem in sensor networks under different communication paradigms and addressed the problem of topology optimization under practical network constraints. In particular, we showed that for a fixed (non-time varying) network with ideal communication links, the class of non-bipartite Ramanujan graphs are essentially optimal, given a constraint on the number of network links. We extended the framework to a noisy communication link scenario, and showed the existence of an optimal number of iterations before taking a decision. Finally, we analyzed a random link failure model in Section 6 and considered the problem of optimal allocation of link SNRs leading to the fastest mss convergence rate under a constraint on the total power or communication cost.

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