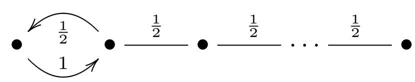
18-799F Algebraic Signal Processing Theory
Spring 2007
Solutions: Assignment 6

(a)

$$\phi(x) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & & & \\ 1 & 0 & \frac{1}{2} & & & \\ 0 & \frac{1}{2} & \ddots & \ddots & & \\ & & \ddots & 0 & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{pmatrix}.$$

The visualization of the signal model is



Unlike the visualization for DFT, this one is bidirectional, has different weights on its edges, and non-cyclic boundary conditions.

(b) In coordinate-free form the Fourier transform is

$$\begin{array}{rrcl} \Delta: & \mathbb{C}/T_n(x) & \rightarrow & \bigoplus_{i=0}^{n-1} (x - \cos\frac{(2k+1)\pi}{2n}) \\ & s & \mapsto & (s(\cos\frac{\pi}{2n}), s(\cos\frac{3\pi}{2n}), \ldots, s(\cos\frac{(2n-1)\pi}{2n})) \end{array}$$

In coordinate form it is

$$\mathcal{F} = \mathcal{P}_{b,\alpha} = [T_l(\alpha_k)]_{0 \le k,l < n} = [\cos\frac{(2k+1)l\pi}{2n}]_{0 \le k,l < n}.$$

In fact, $\mathcal{F}$ is exactly the matrix for $DCT - 3_n$, the Discrete Fourier Transform of type 3.

(c) The frequency response of $h(x) \in \mathcal{A}$ is $(h(\cos\frac{\pi}{2n}), h(\cos\frac{3\pi}{2n}), \ldots, h(\cos\frac{(2n-1)\pi}{2n}))$.

(d) In this signal model the shift is $q = T_1(x)$ and the k-fold shift is $q_k = T_k(q) = T_k(x)$. Using properties $T_{-k} = T_k$ and $T_{n+k} = -T_{n-k}$ for $k < n$ and $T_n = 0$, we can derive corresponding mappings of the basis functions, i.e. shifts, are

$$\phi(T_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\phi(T_1) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$\phi(T_2) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\phi(T_3) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

Then for any $h = h(x) = \sum_{k=0}^{3} h_k T_k \in \mathcal{A}$

$$\phi(h) = \begin{pmatrix} h_0 & \frac{h_1}{2} & \frac{h_2}{2} & \frac{h_3}{2} \\ h_1 & h_0 + \frac{h_2}{2} & \frac{h_1}{2} + \frac{h_3}{2} & \frac{h_2}{2} \\ h_2 & \frac{h_1}{2} + \frac{h_3}{2} & h_0 & \frac{h_1}{2} - \frac{h_3}{2} \\ h_3 & \frac{h_2}{2} & \frac{h_1}{2} - \frac{h_3}{2} & h_0 \end{pmatrix}.$$