18-799F Algebraic Signal Processing Theory
Spring 2007
Solutions: Assignment 4

1. *(40 pts)*

   (a) ϕ is a linear mapping on $\mathbb{C}[x]/p(x)$ because:

      (i) $\mathbb{C}[x]/p(x)$ is a $\mathbb{C}$-vector space ;

      (ii) ϕ is well-defined:

$$\begin{aligned} v(x) \in [q(x)] \quad &\iff \quad p(x)|(q(x)-v(x)) \iff p(x)|x((q(x)-v(x))) \\ &\iff \quad p(x)|(xq(x)-xv(x)) \iff [xv(x)] = [xq(x)]; \end{aligned}$$

      (iii) For any $q(x), v(x) \in \mathbb{C}[x]/p(x)$ and $\alpha, \beta \in \mathbb{C}$:

$$\phi(\alpha q(x) + \beta v(x)) = x(\alpha q(x) + \beta v(x)) = \alpha x q(x) + \beta x v(x) = \alpha\phi(q(x)) + \beta\phi(v(x)).$$

   (b) Since $p(x) = \sum_{i=0}^{n} \beta_i x^i = 0$, then ϕ maps the basis of $\mathbb{C}[x]/p(x)$ as follows:

$$\begin{aligned} 1 \quad &\mapsto \quad x \\ x \quad &\mapsto \quad x^2 \\ &\dots \\ x^{n-1} \quad &\mapsto \quad x^n = \tfrac{p(x)-\sum_{i=0}^{n-1}\beta_i x^i}{b_n} \end{aligned}$$

   Hence, the matrix B_ϕ is

$$B_\phi = \begin{pmatrix} 0 & 0 & \dots & 0 & -\frac{\beta_0}{\beta_n} \\ 1 & 0 & \dots & 0 & -\frac{\beta_1}{\beta_n} \\ 0 & 1 & \dots & 0 & -\frac{\beta_2}{\beta_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\frac{\beta_{n-1}}{\beta_n} \end{pmatrix}$$

   (c) When $p(x) = x^n - 1$, then $\beta_0 = -1, \beta_n = 1, \beta_i = 0$ for $i = 1, \dots, n-1$. In this case B_ϕ becomes a **circular shift** matrix:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

2. *(30 pts)*

   (a) ϕ maps each $x^i, i = 0, \dots, n-1$ to $(\alpha_0^i, \dots, \alpha_{n-1}^i)$. Thus, matrix B_ϕ is

$$\begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \dots & \alpha_0^{n-1} \\ 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \dots & \alpha_{n-1}^{n-1} \end{pmatrix}$$

   (b) If $p(x) = x^n - 1 = \prod_{i=0} n - 1(x - w_n^i)$, where $w_n = e^{\frac{2\pi\sqrt{-1}}{n}}$ is the n-th root of unity, then matrix B_ϕ becomes a $\mathbf{DFT}_n$ matrix:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w_n & w_n^2 & \dots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & \dots & w_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_n^{n-1} & w_n^{2(n-1)} & \dots & w_n^{(n-1)(n-1)} \end{pmatrix}$$

3. *(30 pts)*

(a) Applying the solution from problem 2(b), we immediately get

$$B_\phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

(b)

$$\begin{array}{rrcl} \psi : & \mathbb{C}[x]/(x^4-1) & \rightarrow & \mathbb{C}[x]/(x^2-1) \oplus \mathbb{C}[x]/(x^2+1) \\ & q(x) & \mapsto & (q(x) \mod (x^2-1), q(x) \mod (x^2+1)) \end{array}$$

Then with respect to bases $\{1, x\}$ in both $\mathbb{C}[x]/(x^2-1)$ and $\mathbb{C}[x]/(x^2+1)$, the basis of $\mathbb{C}[x]/(x^4-1)$ is mapped as follows:

$$\begin{array}{rcl} 1 & \mapsto & ((1,0),(1,0)) \\ x & \mapsto & ((0,1),(0,1)) \\ x^2 & \mapsto & ((1,0),(-1,0)) \\ x^3 & \mapsto & ((0,1),(0,-1)) \end{array}$$

The matrix of this mapping is

$$B_\psi = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = DFT_2 \otimes I_2$$

4. *(30 pts)*

(a) Applying the Chinese Remainder Theorem. we get mapping

$$\begin{array}{rrcl} \mu : & \mathbb{C}[x]/(x^2-1) \oplus \mathbb{C}[x]/(x-1) & \rightarrow & \mathbb{C}[x]/(x+1) \oplus \mathbb{C}[x]/(x^2+1) \bigoplus \mathbb{C}[x]/(x-i) \oplus \mathbb{C}[x]/(x+i) \\ & (q(x), r(x)) & \mapsto & (q(1), q(-1), r(i), r(-i)) \end{array}$$

The basis of $\mathbb{C}[x]/(x^2-1) \oplus \mathbb{C}[x]/(x-1)$ is $\{(1,0),(x,0),(0,1),(0,x)\}$. μ maps it as follows:

$$\begin{array}{rcl} (1,0) & \mapsto & (1,1,0,0) \\ (x,0) & \mapsto & (1,-1,0,0) \\ (0,1) & \mapsto & (0,0,1,1) \\ (0,x) & \mapsto & (0,0,i,-i) \end{array}$$

Hence, the matrix of the mapping is

$$B_\mu = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = (I_2 \otimes DFT_2) T_2^4$$

(b) We can decompose mapping ϕ into a composition of mappings ψ and μ. The only problem is the order of the roots of $x^4 - 1$: in case of ϕ it is $1, i, -1, -i$, while for ψ and μ it is $1, -1, i, -i$. We need one more step in the decomposition, namely the permutation that swaps the second and third summands $\mathbb{C}[x]/(x-i)$ and $\mathbb{C}[x]/(x+1)$. The corresponding matrix is called a *permutation matrix*

$$L_2^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combining L^4_2, B_ψ and B_μ, we decompose B_ϕ as

$$\begin{aligned} B_\phi &= L^4_2 B_\mu B_\psi = L^4_2 (I_2 \otimes DFT_2) T^4_2 (DFT_2 \otimes I_2) \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \end{aligned}$$