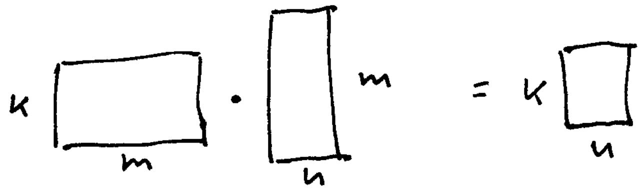
Mat-mat Mult:

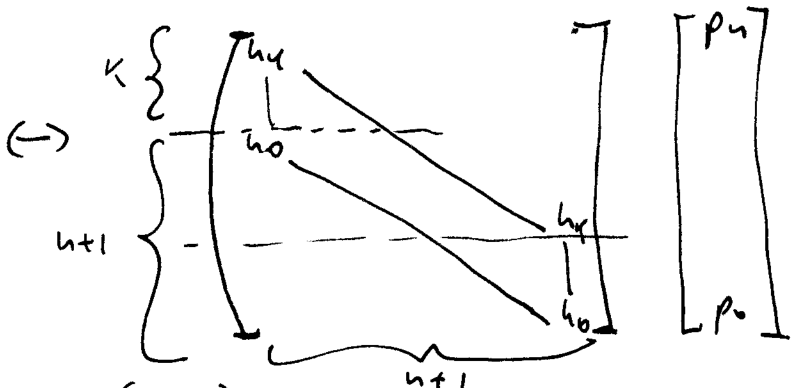


- $O(kmn)$
- More precisely: $(2m-1)\cdot kn$ ops
- Even more: mkn mults, $(m-1)kn$ adds

Poly Mult.

$h(x) = h_k x^k + \dots + h_0, \quad p(x) = p_n x^n + \dots + p_0$

$h(x) = h_k p_n x^{n+k} + (h_{k-1} p_n + h_k p_{n-1}) x^{n+k-1} + \dots + h_0 p_0$



- $O(kn)$
- $(k+1)(n+1)$ mults,

$2(0+1+\dots+k-1) + k(n+1-k)$

$= k(k-1) + kn + k - k^2 = kn$ adds

Exact Recurrences

easy case:

$$f_0 = c$$
$$f_k = a f_{k-1} + s_k, \qquad k \geq 1 \qquad (a \text{ independent of } k)$$

$$\Rightarrow \boxed{f_k = a^k c + \sum_{i=0}^{k-1} a^i s_{k-i}}$$

example: $f_0 = 0,\ f_k = 2f_{k-1} + 3 \cdot 2^{k-1} - 1$

$$\Rightarrow f_k = \sum_{i=0}^{k-1} 2^i (3 \cdot 2^{k-i-1} - 1)$$
$$= \frac{3}{2} k 2^k - 2^k + 1$$

exponential version:

$$g_1 = c \qquad\qquad n = 2^k,\ g_{2^k} = f_k,\ t_{2^k} = s_k$$
$$g_n = a g_{n/2} + t_n$$

$$\Rightarrow \quad f_0 = c$$
$$f_k = a f_{k-1} + s_k$$

solve, translate into n

example: $g_1 = 0$ $\xrightarrow{\text{asymptotic}}$ solution $\Theta(n \log(n))$

$$g_n = 2 g_{n/2} + \frac{3}{2} n - 1$$

$n = 2^k$: $f_0 = 0$

$$f_k = 2 f_{k-1} + \frac{3}{2} 2^k - 1$$

solve: $f_k = \frac{3}{2} k 2^k - 2^k + 1$

translate: $g_n = \frac{3}{2} \log_2(n)\, n - n + 1$

Solving recurrences using generating functions

example: Fibonacci numbers

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 2$$

Definition: $(f_n)_{n\geq 0}$ sequence $\longleftrightarrow$ $\sum_{n\geq 0} f_n x^n = F(x)$
generating function for $(f_n)_{n\geq 0}$

solving the recurrence:

$$f_n = f_{n-1} + f_{n-2}$$

① Multiply by x^n and sum

$$\sum f_n x^n = \sum f_{n-1} x^n + \sum f_{n-2} x^n$$

② Determine beginning of summation (avoid f_i, $i<0$)

$$\sum_{n\geq 2} f_n x^n = \sum_{n\geq 2} f_{n-1} x^n + \sum_{n\geq 2} f_{n-2} x^n$$

③ Translate into equation for $F(x)$
(needs initial values f_0, f_1)

$$F(x) - x = x F(x) + x^2 F(x)$$

④ Solve for $F(x)$

$$F(x) = \frac{x}{1-x-x^2}$$

⑤ Rational Expansion

$$F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\alpha' x}$$

$1/\alpha$, $1/\alpha'$ zeros of $1-x-x^2$ $\Rightarrow$ $\alpha = \frac{1-\sqrt{5}}{2}$, $\alpha' = \frac{1+\sqrt{5}}{2}$

$$F(x) = \frac{A(1-\alpha' x) + B(1-\alpha x)}{1-x-x^2}$$

compare numerators: $A+B=0$, $-\alpha' A - \alpha B = 1$

$$\Rightarrow A = -\sqrt{5}, \quad B = \sqrt{5}$$

⑥ Evolve into series

$$F(x) = -\sqrt{5} \sum \alpha^n x^n + \sqrt{5} \sum (\alpha')^n x^n$$

(7) Read off result

$$\underline{f_n = -\sqrt{5}\,\alpha^n + \sqrt{5}\,(\alpha')^n}$$

Important principle: map problem into another domain, solve there, and map solution back

$$\begin{tikzcd}
\text{recurrence for sequence } (f_n) \arrow[r, "\text{map}"] \arrow[d, dashed, "\text{difficult}"] & \text{equation for } F(x) \text{ } \cancel{\text{series}} \arrow[d, "\text{solve}"] \\
\text{formula for } f_n & \text{series for } F(x) \arrow[l, "\text{map}"']
\end{tikzcd}$$