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Path Resolution by Coherent Averaging: Trading Spatial and Temporal Degrees of Freedom

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Abstract

The multipath passive location problem with wideband source signal is considered. Spatio/temporally based high resolution algorithms are presented, that assume no knowledge of the second order statistics of the emitted signal. With these methods, the restriction on the size of the receiving aperture commonly imposed by spatial only high resolution algorithms is substituted by a global condition on the total number of spatial and temporal degrees of freedom. This approach shows that the frequency contents of the emitted signal is effectively used to compensate for the eventual defficiency in the number of available sensors, extending the number of detectable paths for an array of a given size.

1. Introduction

High resolution methods for detection and direction estimation of multiple sources are widely discussed in the literature. Originally developed for narrowband (NB), not completely correlated sources, extensions to wideband (WB) possibly coherent sources have been proposed.

The coherent problem is particularly challenging. The orthogonality relation upon which these methods are based (that the noise eigenvectors are orthogonal to the eigenvectors corresponding to the directions of the sources) no longer holds. In the extreme case of completely coherent NB sources, all the snapshots have colinear noise-free components, defining a signal subspace of dimension 1, generated by a single linear combination of all the steering vectors.

The paper discusses alternative methods that explore the spatial (number of sensors K) and temporal (time bandwith product N) degrees of freedom in a cooperative fashion to resolve P perfectly coherent paths. The resolvability constraint is

$$K + N > 2P. \qquad (1)$$

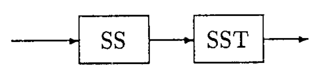
Next, we revisit the spatial smoothing method of [1], presenting an elegant and short proof of their result. Section 3 summarizes a temporal only algorithm ($K = 1$, $N > 2P$) demonstrated in [4]. Section 4 outlines a spatial/temporal procedure with resolvability condition ($N > P$,$N + K > 2P$). Finally, Section 5 describes the dual method ($K > P$,$N + K > 2P$).

2. Spatial Smoothing Revisited

In [1], the NB coherent problem is solved when

$$K \geq 2P. \qquad (2)$$

Spatial smoothing (SS) of the data is followed by a signal subspace technique (SST) as in the scheme:



We now prove that (2) is the resolvability condition for P paths. Reinterpret the noise free component y_k at the output of sensor k of a linear uniform array as the impulse response of the following system:

$$\begin{array}{ll} \mathrm{e}_{k+1} = D\ \mathrm{e}_k + b\ u_k & \text{state eq.} \\ \mathrm{e}_0 = 0 & \\ z_{k+1} = \mathrm{s}^T\ \mathrm{e}_{k+1} & \text{output eq.} \end{array} \qquad (3)$$

where

- $\mathrm{e}_k \in \mathbf{C}^P, u_k \in \mathbf{C}, y_k \in \mathbf{C}$
- $D = \text{diag}\ [e^{j\omega\theta_1} \cdots e^{j\omega\theta_P}]$
- $\mathrm{s} = [s_1 \cdots s_P]$ is the vector of source signals.

When ($u_0 = 1, u_k = 0, k \geq 1$), the state $\mathrm{e}_k \in \mathbf{C}^P$ is the vector of the k-th components of the P steering

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vectors. Define the Hankel pattern associated with the impulse response y_k of system (3):

$$H = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_M \\ y_2 & y_3 & & & y_{M+1} \\ \vdots & \vdots & & & \vdots \\ y_{q-1} & y_q & & & y_{M+q-2} \\ y_q & y_{q+1} & & \cdots & y_{M+q-1} \end{bmatrix} \quad (4)$$

Using the notation of Eqs. (10) and (11) of [2], it is easy to see that

$$A\overline{S}A = \frac{1}{M} H\ H^H. \quad (5)$$

Theorem [1]
rank $H\ H^H = P$ iff

$$\begin{cases} s_i \neq 0, & i = 1, \ldots, P, \\ D_{ii} \neq D_{jj} & i, j = 1, \ldots, P \end{cases} \quad (6)$$

Proof: Matrix (5) has rank P iff H has rank P. By a theorem due to Padé [5], H is full rank iff the system (3) is of minimal dimension. By a standard result of Kalman [6], system (3) has minimal dimension iff

$$\begin{array}{l} \text{rank}\left[\mathbf{b}|D\mathbf{b}|\cdots|D^{P-1}\mathbf{b}\right] = P \\ \text{rank}\left[\mathbf{s}|D\mathbf{s}|\cdots|D^{P-1}\mathbf{s}\right] = P \end{array} \quad (7)$$

These are true iff (6) are true. □

It can be concluded that these methods overcome the defficiency in degrees of freedom of the incoming signal by further spatial processing of the observation vector. Combining the requirements $q > P, M > P$ the condition on the number of sensors is:

$$K > 2P. \quad (8)$$

In the wideband context there are additional degrees of freedom in the temporal dimension of the observations, which can be used to compensate for the perfect coherency of the sources. In [3], a wideband coherent method is presented that uses the frequency diversity of the emitted signal to resolve the source matrix singularity. In the coherent case this method requires at least the minimum bounds:

$$N > P, \qquad K > P \quad (9)$$

where N is the number of independent frequency components of the observations (time-bandwith product). The resolvability condition is given independently in N and K, reflecting the distinct roles assigned to each one by this method: frequency (N) is used to define a nonsingular "source matrix", while space (K) is used to define the signal subspace at each frequency.

Our claim is that temporal (frequency) and spatial degrees of freedom can be interchanged, and thus that a global condition on the total number of degrees of freedom is sufficient.

3. Temporal Method

Let the signal received at the k-th sensor be described by:

$$r_k(t) = \sum_{p=1}^{P} a_{pk}\ s(t - \tau_{pk}) + w_k(t), t \in T, k = 1, \ldots, K \quad (10)$$

where:

- $\{a_{pk}\}_{k=1,p=1}^{K,P}$ and $\{\tau_{pk}\}_{k=1,p=1}^{K,P}$ are the attenuation and delay from the source to sensor k along path p;
- $s(t)$ is the emitted wideband signal, assumed adequately described, over the observation interval, by:

$$s(t) = \sum_{n=1}^{N} s_n\ e^{j\omega_n t}. \quad (11)$$

  The frequencies ω_n are equispaced: $\omega_n - \omega_{n-1} = \Delta$, and s_n are random variables of unknown covariance matrix.
- $w_k(t)$ is the observation noise at sensor k, independent of the signal $s(t)$. The (temporal/spatial) covariance of $w_k(t)$, Σ, is assumed known.

With the assumption (11), the rank of the covariance matrix of the noise free component of the stacked vector of observations:

$$Q = E[x\ x^T], \quad (12)$$

where

$$x^T = \left[\mathbf{r}^T(t_1)|\cdots|\mathbf{r}^T(t_l)\right] \quad (13)$$

is N. The eigenvectors of Q in the metric of the noise, i.e., solutions of the equation:

$$Q\Sigma^{-1}\ u_i = \lambda_i u_i \quad (14)$$

corresponding to the N largest eigenvalues span the same space as the columns of the $(K\ L) \times N$ matrix G, see [4]

$$\begin{aligned} G^T &= [G(t_1)^T \cdots G(t_L)^T] \\ G(t_q)_{kn} &= \sum_{p=1}^{P} a_{kp} f_n(t_q - \tau_{kp}) \\ f_n(t) &= e^{j\omega_n t}. \end{aligned} \quad (15)$$

That is,

$$\text{Span}\{u_1, \ldots, u_n\} = \text{Span}\{\text{columns of } G\}. \quad (16)$$

From (16), there exists an invertible matrix T

$$U = [u_1|\cdots|u_N] = G\,T. \quad (17)$$

It is shown in [4], that the l-th sub-bloc of G is a $L \times N$ matrix Y_l such that:

$$Y_l^T = F(t_l)A \quad (18)$$

where $F(t_l)$ depends on the known functions $f_n(t)$ and on the delays τ_{kp}, and A is a block diagonal matrix with $P \times 1$ entries:

$$A_{kk} = \mathbf{a}_k = [a_{k1} \cdots a_{kP}]^T. \quad (19)$$

The k-th column of Y_l^T depends only on the propagation parameters corresponding to the k-th sensor:

$$\left[Y_l^T\right]_k = F_k(t_l)\mathbf{a}_k \quad (20)$$

where

$$F_k(t_l) = [\mathbf{v}(t_l, \tau_{k1})|\cdots|\mathbf{v}(t_l, \tau_{kP})]\mathbf{a}_k \quad (21)$$

and

$$\mathbf{v}(t_l, \tau_{kp}) = [f_1(t_l - \tau_{kp})|\cdots|f_N(t_l - \tau_{kp})]^T. \quad (22)$$

Since the frequencies ω_n are equispaced, $\mathbf{v}(t_l, \tau_{kp})$ are Vandermonde vectors. We apply to them the "spatial smoothing" technique [1], defining the vectors:

$$y_{l,k}^m = [[Y_l^T]_{km} \cdots [Y_l^T]_{km+q-1}]^T, m = 1, \ldots, M. \quad (23)$$

These vectors can be utilized to define a signal subspace at sensor k. The "covariance matrix" defining these signal subspaces is given by:

$$R_k = F_k S_k F_k^H \quad (24)$$

where S_k is a nonsingular $P \times P$ matrix, and the matrix F_k has columns:

$$\mathbf{v}(\tau_{kp}) = \left[\mathbf{v}_q(t_1, \tau_{kp})^T|\cdots|\mathbf{v}_q(t_N, \tau_{kp})^T\right]^T \quad (25)$$

where $\mathbf{v}_q(t_l, \tau_{kp})$ is formed by the first q components of the vector $\mathbf{v}(t_l, \tau_{kp})$ (see (22)). Application of a conventional eigendecomposition method to R_k yields estimates for P and for the delays τ_{kp}. The role of the steering vectors in the NB spatial methods is played here by the vectors $\mathbf{v}(\tau_{kp})$, dependent on the known functions $f_n(t)$, and the unknown parameters τ_{kp}.

The matrix T depends on the values of the parameters that are being estimated. In [4], this algorithm is recursive, with initial values provided by some other method. The resolvability condition, derived from the requirements $q > P$ and $M > P$, is

$$N > 2P \quad (26)$$

dual of the condition for NB coherent methods: $K > 2P$. Eq.(26) says that P paths can be resolved even in the extreme case of one sensor $K = 1$.

4. Temporal/Spatial Method

The method proposed in this section incorporates knowledge about the spatial structure of the observed field into the temporal algorithm of section 3, finding a global signal subspace for the entire array.

Analogously to the coherent signal subspace method of [3], we define transformation matrices, L_k, that map the signal subspace at sensor k into the signal subspace of a reference sensor k_0:

$$L_k\, F_k = F_{k_0}. \quad (27)$$

The coherently averaged matrix W:

$$\begin{aligned} W &= \sum_{k=1}^{K} L_k\, V_k\, L_k^H \\ &= L_{k_0}\left[\sum_{k=1}^{K} S_k\right] F_{k_0}^H \end{aligned} \quad (28)$$

will have rank P as long as the rank of

$$S = \sum_{k=1}^{K} S_k \quad (29)$$

is P. The condition on M ($M > P$) in force in the temporal only method of the previous section can now be relaxed. For this combined temporal/spatial approach the source matrix is

$$\overline{S} = \sum_{k=1}^{K}\sum_{m=1}^{M} D_k^{(m-1)} a_k a_k^H D_k^{(m-1)^H} \quad (30)$$

where

$$D_k = \text{diag}\{e^{j\Delta\tau_{k1}} \cdots e^{j\Delta\tau_{kP}}\} \quad (31)$$

and Δ is the frequency spacing.

In [4], for a uniform linear array,

$$D_k = D_0 D^{k-1} \quad (32)$$

with

$$D = \mathrm{diag}\{e^{j\Delta d_1} \dots e^{j\Delta d_P}\} \qquad (33)$$
$$D_0 = \mathrm{diag}\{e^{j\Delta d_{01}} \dots e^{j\Delta d_{0P}}\}$$

where d_p is the intersensor delay for the signal travelling along path p and d_{0p} the phase relative to a reference point. From (32) and (30),

$$\overline{S} = \sum_{k=1}^{K}\sum_{m=1}^{M}(D^{(k-1)}D_0)^{(m-1)}a_k a_k^H(D^{(k-1)}D_0)^{(m-1)^H}. \qquad (34)$$

The application of a conventional high resolution method to W to successively resolve the P paths requires that matrix (34) be nonsingular.

Although it is evident that whenever either $K > P$ or $M > P$, $\overline{S}$ has full rank, this last equation also suggests that a combined condition in K and M must exists that ensures the nonsingularity of $\overline{S}$.

The study of the rank of matrix (34) can be shown to be equivalent to the study of a system of homogeneous multivariate polynomials [4]. The analytic study of that system for general values of P, K and M presents major difficulties. For the tested values of P (only small values of P were studied) the condition:

$$N + K > 2P \qquad (35)$$

derived from $M + K > P$ and $q > P$, actually yields a full rank matrix $\overline{S}$, except possibly over a set of arrival angles of measure zero.

The algorithm can be partitioned in the following steps:

1. The vectors $[Y_l]_k$ that are a linear combination of the vectors $v(\tau_{kp})$, of the known model, are determined from the "signal eigenvalues" of Q .
2. Frequency smoothing is applied, increasing the rank of the "source matrix" at each sensor.
3. Finally, the spatial structure is used to combine the signals at each sensor, by means of the matrices L_k.

5. Spatial/Temporal Method

A dual method is now briefly formulated. This method starts by calculating the signal subspace at each frequency of the incoming waveform. Then, using the fact that $K > P$, spatial smoothing is applied to each frequency component of the incoming waveform. Finally, transformation matrices map the signal subspace at each frequency into the signal subspace at a given frequency, and are then combined, giving the global signal subspace. When the spatial smoothing step is eliminated in this dual method, it coincides with the WB coherent method of [3] .

In [3] the resolvability constraint is only stated asymptotically, as

$$K > P \qquad N \text{ large.} \qquad (36)$$

For the completely coherent case, it is clear that the technique of [3] requires at least $N > P$. It is interesting to see that incorporation of just one step of spatial smoothing ($M = 2$) is sufficient to find more accurate bounds. For equally spaced frequencies the following conditions ensure a nonsingular $\overline{S}$:

$$K > P + 1 \qquad N \geq P. \qquad (37)$$

The source matrix becomes

$$\overline{S} = \sum_{n=1}^{N}[S(\omega_n) + (D^{(n-1)}D_0)S(\omega_n)(D^{(n-1)}D_0)^H]. \qquad (38)$$

The recursion in n guarantees, with the bounds (37), that $\sum_{n=1}^{N}(D^{(n-1)}D_0)S(\omega_n)(D^{(n-1)}D_0)^H$ is a definite positive matrix. Since $\sum_{n=1}^{N} S(\omega_n)$ is semidefinite positive, their sum must be definite positive, ensuring thus the nonsingularity of $\overline{S}$.

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