OCEAN ACOUSTIC TOMOGRAPHY STRUCTURED COVARIANCE ESTIMATION

Sébastien Bausson [1*], *José M. F. Moura* [2], *and Didier Mauuary* [1]

[1] Laboratoire des Images et des Signaux, ENSIEG, INPG, BP46, 38402 Grenoble, France
[2] Carnegie Mellon University, Dep. of Electrical and Computer Eng., Pittsburgh, PA 15213, USA

ABSTRACT

Classic Ocean Acoustic Tomography by Wiener inversion needs good estimates of the noises power affecting the errors between the *in situ* measurements of the travel times and their estimates obtained by reliable simulations. We investigate the maximum likelihood estimation of a structured covariance matrix, whose subspaces of interest are known, but whose associated powers are unknown. Using the Ocean Acoustic Tomography constraints, we assume that the covariance is the sum of a full rank known matrix and an unknown component. We derive the maximum likelihood estimates for these noise powers and compute the Fisher information matrix to get insight into the geometric properties of the estimators. We verify with a realistic classic Ocean Acoustic Tomography simulation the good quality of our noise power estimates.

1. INTRODUCTION

Classic Ocean Acoustic Tomography (OAT) is an inverse method to map sound velocity and current fields in the ocean. Twenty years of development work provide us with an ocean acoustic propagation atlas ([1], p382-401) and with reliable oceanic models. Single slice OAT gives only average information along the ray path structure, and is restricted to deep ocean with no bottom and surface interactions. To invert shallow water channels, it is important to combine OAT, which provides constraint information –prior model– with ocean measurements –data assimilation– that are now cheaply available with inexpensive oceanographic instrumentation (Temperature, Conductivity, Depth sensors).

Data assimilation relaxes the hard constaint of accurate tracking the position of the sensors. Reference [2], for example, assumes that with large planar array of sensors the OAT inverse operator is insensitive to sensor motion as long as one has a good estimate of the power of the errors, including position errors, clock driffting, or ambient acoustic noise. This paper addresses the estimation of the noise powers, casting this problem as a special case of the general structured covariance estimation in the linear statistical model [3, 4, 5]. In OAT, it is often possible to derive from historical data and first principles reasonable estimates of the structure of the underlying signals and noises subspaces. The structured covariance estimate reduces to finding the power parameters of the components of these subspaces.

*The first author performed the work when visiting CMU

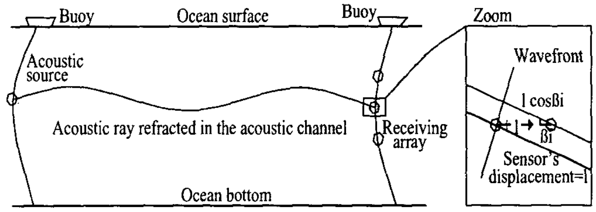

Fig. 1. Ray propagation and sensor displacement offset.

In section 2, we introduce the classic tomography model. In section 3, we derive the Maximum Likelihood (ML) regression equations for the constrained parameter estimates and the Cramér-Rao bound. In section 4, we apply these results to a realistic simulation of classic OAT for a single couple source/receiver setup.

2. TIME-OF-FLIGHT TOMOGRAPHY

Classic OAT infers from measurements of the pulses travel time the state of the ocean traversed by a sound field. This contrasts with matched-field OAT that uses the whole acoustic pressure field. We present here the linear statistical model corresponding to the forward step in classical OAT.

The travel time $\tau_i(t)$ along ray R_i from source to receiver in the sound velocity field $C(x, y, z, t)$ is:

$$\tau_i(t) = \int_{R_i} \frac{ds}{C(x, y, z, t)} + n_i(t),$$

where s is the arc length along the ray path R_i, assumed known and fixed, *i.e.*, s depends on x,y,z but not t, and n_i is a general error term. Ocean sound speed variations are linearized around a nominal sound velocity field $C_0(x, y, z)$ estimated from the historical data. To first order, the expression for the travel time perturbation term becomes:

$$\delta\tau_i(t) = \int_{R_i} \frac{-\delta C(x,y,z,t)ds}{C_0^2(x,y,z)} + \frac{\cos(\beta_i)l(t)}{C_0(x_s,y_s,z_s)} + n_i(t), \quad (1)$$

where β_i is the angle between the locally linear wavefront i and the horizontal as in figure 1 at the source position x_s, y_s, z_s. The vertical variation of the sensor is assumed to be corrected with a pressure meter that gives the depth. For a complete model see [6]. All rays measured at the snapshot t are biased by the same displacement $l(t)$.

To describe the ocean perturbation model, it is common to project the perturbation to an eigenvector basis – the Empirical Orthogonal Functions (EOF). These EOF are the eigenvectors of the velocity field perturbation covariance matrix that is estimated from historical data. Write the perturbation $\delta C(x,y,z,t)$ on the EOF basis $\{U_k\}$ as

$$\delta C(x,y,z,t) \simeq \sum_{k=1}^{p} \theta_k(t) U_k(x,y,z), \quad (2)$$

where p, the number of EOFs to be kept in the inversion, depends on the quality of the historical data. We group the measured $\delta\tau_i$, the unknown θ_k, and the noise n_i in vectors $\delta\tau$, θ, and n of dimension N, p, and N respectively. The symbol N is the number of rays. Let H and g be the $N \times p$ matrix and the N-dimensional vector that collect the EOF and the mooring motion structure. From eqs. (1) and (2):

$$\delta\tau(t) = H\theta(t) + gl(t) + n(t).$$

Assuming θ, l, and n are multivariate normal distributions of zero mean and covariance Γ_θ, σ_l^2, and $\sigma_n^2 I_N$, we get the structured covariance matrix:

$$R = < \delta\tau\delta\tau^t > = H\Gamma_\theta H^t + \sigma_l^2 gg^t + \sigma_n^2 I_N, \quad (3)$$

where H and g are known. We consider a more general structured covariance than (3), namely, we extend it to:

$$R = R_o + \sum_{k=1}^{m_l} \sigma_{l_k}^2 G_k G_k^t, \quad (4)$$

where $R_o = H\Gamma_\theta H^t + \sigma_n^2 I_N$ is a $N \times N$ matrix and G_k are full rank $N \times r_k$ matrices. In other words: we assume that we may have several sources of structured errors –m_l sources– that we know the subspace structure of each of these error sources –the column spaces of G_k, $1 \le k \le m_l$– but that the relative strength of these error sources – the parameters $\sigma_{l_k}^2$, $1 \le k \le m_l$– are unknown. In this paper we further assume that R_o is full rank and known, which is equivalent to assuming that Γ_θ and σ_n^2 are known. This is reasonable in OAT where an iterative procedure is used. At the first step of the iteration, we approximate Γ_θ by the eigenvalues of the ocean perturbation correlation matrix. We approximate σ_n^2 by the average of the lowest eigenvalues of the covariance matrix R. The estimated $\widehat{\theta}_k(t)$ are used to correct the ocean model –the Γ_θ and the simulated travel time arrivals– before running another inversion.

3. ML ESTIMATION

To find the maximum likelihood (ML) estimator $\widehat{\sigma_{l_k}^2}$ of $\sigma_{l_k}^2$, we root the system of regression equations ([5], p260):

$$\text{tr}\left\{R^{-1}(R-S)R^{-1}\frac{\partial R}{\partial \sigma_{l_k}^2}\right\} = 0, \qquad k = 1,..,m_l, \quad (5)$$

where tr is the trace and $S = \frac{1}{m}\sum_{i=1}^{m} \delta\tau_i \delta\tau_i^t$ is the sample covariance. The element of the Fisher information matrix (FIM) for the joint estimation of $\sigma_{l_k}^2$ and $\sigma_{l_j}^2$ is:

$$J_{kj} = \frac{m}{2}\text{tr}\left\{R^{-1}\frac{\partial R}{\partial \sigma_{l_k}^2}R^{-1}\frac{\partial R}{\partial \sigma_{l_j}^2}\right\}. \quad (6)$$

3.1. Inversion matrix lemma

To work further with equations (5) and (6) we need the following technical lemma.

Lemma 1 *Let $A_{N\times N}$, $B_{N\times k}$, $C_{k\times k}$, and $D_{k\times N}$ be four matrices. If A, C, and $R = A + BCD$ are non singular, then*

$$D\left[A + BCD\right]^{-1} = \left[CDA^{-1}B + I\right]^{-1} DA^{-1}. \quad (7)$$

Proof. Start with the inverse of a small-rank adjustment ([7], p19)

$$R^{-1} = A^{-1} - A^{-1}B\left[DA^{-1}B + C^{-1}\right]^{-1} DA^{-1}.$$

Premultiplying by D and factoring on the right DA^{-1}

$$DR^{-1} = \left\{I - DA^{-1}B\left[DA^{-1}B + C^{-1}\right]^{-1}\right\} DA^{-1}.$$

Finally, factor on the right $\left[DA^{-1}B + C^{-1}\right]^{-1}$

$$DR^{-1} = C^{-1}\left[DA^{-1}B + C^{-1}\right]^{-1} DA^{-1},$$

from which the lemma follows.

3.2. ML estimation of the subspace power parameter

For given k, $1 \le k \le m_l$, rewrite eq. (4) as $R = R_{\overline{k}} + \sigma_{l_k}^2 G_k G_k^t$. Since $R_{\overline{k}}$ is invertible and from the lemma:

$$G_k^t R^{-1} = \left[\sigma_{l_k}^2 G_k^t R_{\overline{k}}^{-1} G_k + I_{r_k}\right]^{-1} G_k^t R_{\overline{k}}^{-1},$$

from which, since G_k is full rank,

$$G_k^t R^{-1} = \left[\sigma_{l_k}^2 I_{r_k} + (G_k^t R_{\overline{k}}^{-1} G_k)^{-1}\right]^{-1} D_k^t,$$

where D_k is the $N \times r_k$ matrix

$$D_k^t = (G_k^t R_{\overline{k}}^{-1} G_k)^{-1} G_k^t R_{\overline{k}}^{-1} = (R_{\overline{k}}^{-t/2} G_k)^{\#} R_{\overline{k}}^{-t/2},$$

where $M^{\#}$ is the pseudo inverse of the $N \times r$ matrix M of rank r: $M^{\#} = (M^t M)^{-1} M^t$, and $R_{\overline{k}}^{-1/2}$ is the inverse of the upper triangular Cholesky factor of $R_{\bar{k}}$, *i.e.*, $R_{\bar{k}} = R_{\bar{k}}^{t/2} R_{\bar{k}}^{1/2}$. It follows that $(G_k^t R_{\bar{k}}^{-1} G_k)^{-1} = D_k^t R_{\bar{k}} D_k$ and

$$G_k^t R^{-1} = (D_k^t R D_k)^{-1} D_k^t, \quad (8)$$

since $D_k^t G_k G_k^t D_k = I_{r_k}$.

Regression equation The derivative term in eq. (5) is $G_k G_k^t$. Using the trace property $\text{tr}(AB) = \text{tr}(BA)$, eq. (5) becomes after manipulation

$$\text{tr}\left\{(D_k^t R D_k)^{-2}\left[D_k^t(R-S)D_k\right]\right\} = 0,\ k=1,..,m_l. \quad (9)$$

Fisher information matrix A similar derivation leads to the FIM generic element

$$\begin{aligned} J_{kk} &= \frac{m}{2}\text{tr}\left\{(D_k^t R D_k)^{-2}\right\} \quad k,i = 1,..,m_l \quad (10) \\ J_{ki} &= \frac{m}{2}\text{tr}\left\{(D_k^t R D_k)^{-2} D_k^t G_i G_i^t D_k\right\}. \end{aligned}$$

Replacing R and D_k by their definition in J_{kk} from eq. (10)

$$J_{kk} = \frac{m}{2}\text{tr}\left\{\left[\sigma_{l_k}^2 I_{r_k} + (G_k^t R_{\bar{k}}^{-1} G_k)^{-1}\right]^{-2}\right\}. \quad (11)$$

Let the singular value decomposition of $A_{\bar{k}}^{-1} = R_{\bar{k}}^{-t/2} G_k = U\Lambda^{-1}V^t$. The diagonal matrix Λ is of dimension r_k, set $\Lambda(j,j) = \sigma_{A_k j}$. Eq. (11) becomes

$$J_{kk} = \frac{m}{2}\sum_{j=1}^{r_k}\left[\sigma_{l_k}^2 + \sigma_{A_k j}^2\right]^{-2} \qquad k = 1,..,m_l.$$

The FIM entry J_{kk}, $1 \le k \le m_l$, is small if the singular values $\sigma_{A_k j}^{-1}$ are small for all j, $1 \le j \le r_k$. These singular values are the lengths of the semi-axes of the hyperellipsoid associated with G_k after its projection on $R_{\bar{k}}^{-1/2}$.

Cramér-Rao bound The variance of any unbiased estimator is lower bounded by the diagonal entry of the CRB, which itself is lower bounded by the inverse of the corresponding entry in the FIM ([5] p231)

$$< (\widehat{\sigma_{l_k}^2} - \sigma_{l_k}^2)^2 > = \sigma_{ek}^2 \ge J^{-1}(k,k) \ge J_{kk}^{-1} \quad k = 1,..,m_l.$$

The variance bound of the error of the power estimates depends on the projection of the associated error subspaces on the other subspaces in the covariance matrix.

Special case 1: at least one subspace is rank one Assume that one of the error subspaces, say subspace k, is rank one, and represent G_k and D_k by g_k and d_k. Then, from eq.(9), we get the ML estimate of $\sigma_{l_k}^2$

$$\widehat{\sigma_{l_k}^2} = d_k^t(S - R_{\bar{k}})d_k, \quad d_k^t = (R_{\bar{k}}^{-t/2} g_k)^{\#} R_{\bar{k}}^{-t/2}. \quad (12)$$

This reduces by one dimension the search algorithm for the other unknowns. The FIM is

$$J_{kk} = \frac{m}{2}(d_k^t R d_k)^{-2}, \qquad J_{ki} = J_{kk}\, d_k^t G_i G_i^t d_k. \quad (13)$$

The remaining J_{ii}'s are given by eq.(10). Let $R_{\bar{k}} = U\Sigma_{\bar{k}}U^t$ be the eigenvalue decomposition. Denote by $\sigma_{\bar{k}j}^2$ the diagonal entries of $\Sigma_{\bar{k}}$, and by u_j the vector $U(:,j)$ then

$$\sigma_{ek}^2 \ge J_{kk}^{-1} = \frac{2}{m}\left[\sigma_{l_k}^2 + \left(\sum_{j=1}^{N}\sigma_{\bar{k}j}^{-2}(g_k^t u_j)^2\right)^{-1}\right]^2, \quad (14)$$

from which a geometric interpretation follows. The largest eigenvalues $\sigma_{\bar{k}j}^2$ are associated with the eigenvectors u_j corresponding to the error subspaces or signal signatures in $R_{\bar{k}}$. If g_k is orthogonal to the first group of eigenvectors of $R_{\bar{k}}$, the sum term in eq. (14) will be large and the lower bound will be small. The variance bound of the error of the rank one subspace g_k power estimate is small when g_k is orthogonal to the subspace defined by the signal and the other error sources.

After expanding R in eq. (13), we obtain from J_{kk}^{-1}

$$\sigma_{ek}^2 \ge \frac{2}{m}\left[\|d_k^t R_o^{t/2}\|^2 + \sigma_{l_k}^2 + \sum_{i=1,i\neq k}^{m_l}\sigma_{l_i}^2\|d_k^t G_i\|^2\right]^2.$$

For a generic matrix M let $\tilde{M} = R_{\bar{k}}^{-t/2}M$. The term $d_k^t G_i$ is then equivalent to $\tilde{g}_k^{\#}\tilde{G}_i$. The norm of this vector is a measure of the colinearity of g_k and G_i weighted by $R_{\bar{k}}$. This last expression shows a partial separation of the influence of the signal in R_o from that of the subspaces of the error sources subspaces.

Special case 2: at least two subspaces are rank one Assume that two subspaces, say subspaces k and q, $1 \le k < q \le m_l$, are rank one and represent G_k and G_q by g_k and g_q. Note $R = R_0 + \sigma_{l_k}^2 g_k g_k^t + \sigma_{l_q}^2 g_q g_q^t$ and use Woodbury's identity several times on eq. (13) to get the FIM principal sub-matrix corresponding to the power estimation of the k and q error subspaces. The determinant of this sub-matrix may be expressed with $a_k = g_k^t R_0^{-1} g_k$, $a_q = g_q^t R_0^{-1} g_q$, and $a_{kq} = g_k^t R_0^{-1} g_q = (a_k a_q)^{1/2} cos\phi$ where ϕ is the angle between the vectors $R_0^{-t/2} g_k$ and $R_0^{-t/2} g_q$ as:

$$\begin{aligned} \det(J_{k,q}) &= \frac{m^2}{4} D^{-3} C\left[DC + 2a_{kq}^2\right], \\ D &= \left(1 + \sigma_{l_k}^2 a_k + \sigma_{l_q}^2 a_q + \sigma_{l_k}^2\sigma_{l_q}^2 C\right), \\ \text{and} \quad C &= a_k a_q - a_{kq}^2 = a_k a_q sin^2\phi. \end{aligned}$$

If the angle ϕ is zero, the FIM is singular and the CRB on the power estimation for error k and q are infinity. The CRB on the power estimation for the two rank one subspaces are sensitive to the angle between them weighted by the inverse correlation matrix of the other variables.

3.3. Noise powers in classic OAT

The covariance matrix in the OAT setup is in eq. (3). The estimate of the noise power σ_n^2 cannot follow the procedure described previously because $R_{\overline{n}} = H\Gamma_\theta H^t + \sigma_l^2 gg^t$ is singular. We replace $\widehat{\sigma_l^2}$ by its expression, see eq. (12), in $R_{\overline{n}}$. The derivative term in eq. (5) for σ_n^2 is I_N. Diagonalizing $R_{\overline{n}} = U\Delta U^t$, we get

$$\mathrm{tr}\left\{ \left[\widehat{\sigma_n^2} I_N + \Delta\right]^{-2} \left[\widehat{\sigma_n^2} I_N + U^t(R_{\overline{n}} - S)U\right] \right\} = 0.$$

This reduces the problem to a nonlinear equation of one unknown. A root finding algorithm gives an estimate of $\widehat{\sigma_n^2}$ that can be used to find $\widehat{\sigma_l^2}$ given by eq. (12).

4. OAT SIMULATION

We simulate the acoustic transmission from a single couple/receiver with partially known positions, separated by 74km, to study the tomographic reconstruction of a range independent ocean sound velocity profile. The input data are the North-East Atlantic ocean parameters, temperature and salinity converted to sound speed, computed by a high resolution dynamical model, DYNAMO [8] developed at LEGI[1], as integrated in [9]. The inversion estimates the sound speed parameters θ_k, $k = 1, .., p$, for every day during the Summer of 1989. The quality of the reconstruction of the seasonal ocean variability is judged by the sum of the mean square errors (MSE) of the $\theta_k(t)$ estimates. During the simulation, we vary the standard deviation σ_l of the error on the sensors position from 10 cm (essentially precise positionning of the sensors) to 1 km (relatively large positionning error).

Figure 2 compares the sum of the MSE for the estimates of all the $\widehat{\theta_k(t)}$ versus σ_l for four methods. The * plot shows the results with an oblique projection estimation [10]. The remaining plots correspond to three Wiener inversions. The o plot displays the results using perfect weighting functions, *i.e.*, using exact values of σ_l and σ_n. It is a lower bound for the reconstruction MSE. The x plot shows the results using $\widehat{\sigma_n}$ as the average of the lowest singular values of the covariance R and constant σ_l of 30 m. The o curve is close to the x curve at about 30 m, as expected, but blows up away from this. Finally, the + plot shows the result when the Wiener filter uses the estimates of the noise powers provided by the method described in subsection 3.3. The similarity of the o and + plots confirms the good quality of the Wiener inversion using the ML estimates.

[1]Laboratoire des Ecoulements Géophysiques et Industriels, INPG, Grenoble, France

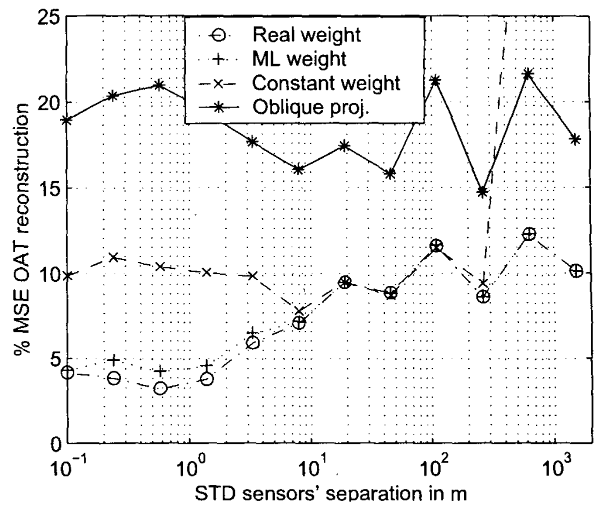

Fig. 2. MSE of OAT reconstruction by four inversions

5. REFERENCES

[1] W. Munk, P. Worcester, and C. Wunsch, *Ocean Acoustic Tomography*, Cambridge University Press, 1995.

[2] T. F. Duda, R. A. Pawlowicz, J. F. Lynch, and B. D. Cornuelle, "Simulated tomographic reconstruction of ocean features using drifting acoustic receivers and a navigated source," *Journal of the Acoustical Society of America*, vol. 98, no. 4, pp. 2270–2279, October 1995.

[3] J. P. Burg, D. G. Luenberger, and D. L. Wenger, "Estimation of structured covariance matrices," *Proceedings of the IEEE*, vol. 70, no. 9, pp. 963–974, September 1982.

[4] F. C. Robey and D. R. Fuhrmann, "Structured covariance estimation via maximal representation of convex sets," in *ICASSP 91, IEEE*, 1991, vol. 5, pp. 3249–3252.

[5] L. L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*, Addison-Wesley Publishing Company, 1991.

[6] B. D. Cornuelle, "Simulations of acoustic tomography array performance with untracked or drifting sources and receivers," *Journal of Geophysical Research*, vol. 90, no. C5, pp. 9079–9088, September 1985.

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[8] The Dynamo Group, "Dynamics of North Atlantic models: simulation and assimilation with high resolution models," *Mast II CT 93-0060.*

[9] S. Bausson and D. Mauuary, "Simulation of ocean acoustic tomography in the North-Eastern Atlantic," in *European Conference on Underwater Acoustic*, July 2000.

[10] R. T. Behrens and L. L. Scharf, "Signal processing applications of oblique projection operators," *IEEE Transactions on Signal Processing*, vol. 42, no. 6, pp. 1413–1424, June 1994.