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LMI Approach to Robust Model Predictive Control[1]

D. Jia,[2] B. H. Krogh,[3] and O. Stursberg[4]

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Abstract. This paper introduces a new approach to robust model predictive control (MPC) based on conservative approximations to semi-infinite optimization using linear matrix inequalities (LMIs). The method applies to problems with convex quadratic costs, linear and convex quadratic constraints, and linear predictive models with bounded uncertainty. If the MPC optimization problem is feasible at the initial control step (the first application of the MPC optimization), it is shown that the MPC optimization problems will be feasible at all future time steps and that the controlled system will be closed-loop stable. The method is illustrated with a solenoid control example.

Key Words. Robust model predictive control, min–max optimization, semi-infinite programming, linear matrix inequalities.

1. Introduction

This paper presents a new approach to robust model predictive control (MPC). In MPC, an optimal control problem is solved at each control step over a specified prediction horizon. The first step in the control input solving the MPC optimization problem is applied to the system and

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[2]Postdoctoral Research Associate, Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, Pennsylvania.
[3]Professor, Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, Pennsylvania.
[4]Senior Researcher, Process Control Laboratory, University of Dortmund, Dortmund, Germany.

then the process is repeated at subsequent control steps. MPC has enjoyed many successful applications because it is an optimal control heuristics that incorporates explicitly the operating constraints (Refs. 1–4). The theoretical issues to be addressed for MPC strategies are to demonstrate that: (i) the MPC optimization problems are feasible at each control step; and (ii) the state is driven to a target set in finite time and remains in the target set indefinitely in the presence of bounded disturbances. For brevity, we refer to the second property as stability.

Robust MPC refers to MPC strategies that use min–max optimization to guarantee that the system operating constraints are satisfied for all possible values of the unknown model parameters and disturbance inputs (Refs. 5–8). An open-loop MPC optimization problem can be too conservative. That is, it may be overlyrestrictive to impose the system constraints for all values of the uncertain parameters for a single input sequence, since this does not reflect the fact that future MPC optimization problems will use the measured state at each control step. To obtain a less conservative solution, MPC optimization problems are formulated in terms of an explicit parametrized feedback control law (Refs. 6, 7, 9, 10).

The robust MPC optimization problem is a semi-infinite optimization problem, since the uncertain parameters vary over infinite sets. The major challenge is to find an effective way to solve the semi-infinite min-max optimization problems at each control step. One approach is to approximate the infinite set of values for the uncertainties by a finite set of values, thereby replacing the infinite number of constraints with a finite number of constraints (Ref. 11). This approach may be exact when the constraint sets are convex polyhedra and the dynamics is linear, since in this case only the extreme points in the uncertain parameter sets need to be considered (Ref. 6).

We propose an alternative approach. Rather than sampling the space of uncertain parameters, we introduce linear matrix inequalities (LMIs) as conservative approximations to the robust system constraints, following the approach suggested in Ref. 12 for general robust optimization problems. The resulting convex optimization problems with LMI constraints can be solved using existing numerical routines (Refs. 13, 14).

Section 2 formulates the optimization problem for the robust MPC. Section 3 introduces a three-step method for solving the robust MPC optimization problem and Section 4 presents the solution method using convex optimizations with LMI constraints. Section 5 shows that all the optimization are feasible and the proposed approach leads to closed-loop stability. A solenoid control example in Section 6 illustrates the approach.

2. Robust MPC Optimization Formulation

For the predictive model in the MPC optimization, we use linear time-invariant (LTI) state equations of the form

$$x_{j+1} = Ax_j + Bu_j + Ev_j, \tag{1}$$

where $x_j \in \Re^{n_x}$ and $u_j \in \Re^{n_u}$ are the state and control respectively and $v_j \in \Re^{n_v}$ denotes the uncertainty in the system model[5]; A, B, E are matrices of appropriate dimensions. We adopt the following time-varying affine state feedback control law in the MPC optimizations:

$$u_j = F_j x_j + u_j^0, \quad j = 1, \ldots, N-1, \quad u_0 = u_0^0, \tag{2}$$

where N is the prediction horizon, $F_j \in \Re^{n_u \times n_x}$, $j = 1, 2, \ldots, N-1$, and $u_j^0 \in \Re^{n_u}$, $j = 0, 1, \ldots, N-1$, are parameters subject to optimization.

Let the sets of constraints on the states and disturbances be denoted by $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N$ and $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{N-1}$ respectively; let $g(x_j, x_{j+1}, u_j, v_j) \leq 0$, where $g: \Re^{n_x} \times \Re^{n_x} \times \Re^{n_u} \times \Re^{n_v} \to \Re^{n_g}$, denote the system operating constraints on the state and control. We assume that the state constraints and the disturbance constraints are ellipsoids, i.e.,

$$\mathcal{X}_{j+1} = \mathcal{E}(Q_{j+1}^x, c_{j+1}^x), \quad \mathcal{V}_j = \mathcal{E}(Q_j^v, c_j^v), \quad j = 0, 1, \ldots, N-1,$$

where

$$\mathcal{E}(Q, c) = \left\{ x \in \Re^n \, \| x - c \|_Q^2 \leq 1 \right\}$$

and $\| \cdot \|_Q$ is the weighted 2-norm of a vector.

For a prediction horizon N, the robust MPC optimization problem at control step k, with the measured state $x(k)$ and decision variables $\mathcal{F} = \{F_1, F_2, \ldots, F_{N-1}\}$, $\mathcal{U} = \{u_0^0, u_1^0, \ldots, u_{N-1}^0\}$, $V = \{v_0, v_1, \ldots, v_{N-1}\}$, is

[5]We distinguish variables for real time from variables in the time horizon in MPC optimization problems by using parentheses for the former [e.g. $x(k)$] and subscripts for the latter (e.g. x_k). Indices for the components of vectors are denoted by superscripts.

formulated as follows:

$$\text{(P1)} \quad \min_{\mathcal{F},\mathcal{U}} \max_{V} \sum_{j=1}^{N} \|x_j\|^2_{\Gamma^x_j} + \sum_{j=0}^{N-1} \|u_j\|^2_{\Gamma^u_j}, \tag{3}$$

subject to

$x_{j+1} = Ax_j + Bu_j + Ev_j,\ j = 0, 1, \ldots, N-1,$ with $x_0 = x(k)$,
$u_j = F_j x_j + u^0_j,\ j = 1, 2, \ldots, N-1, \quad u_0 = u^0_0$,
$v_j \in \mathcal{V}_j = \mathcal{E}(Q^v_j, c^v_j), \quad j = 0, 1, \ldots, N-1,$
for all robustness constraints $\tilde{v}_j \in \mathcal{V}_j,\ j = 0, 1, \ldots, N-1,$

$$\tilde{x}_{j+1} \in \mathcal{X}_{j+1} = \mathcal{E}(Q^x_{j+1}, c^x_{j+1}), \tag{4}$$

$$g(\tilde{x}_j, \tilde{x}_{j+1}, \tilde{u}_j, \tilde{v}_j) \le 0, \tag{5}$$

where

$\tilde{x}_{j+1} = A\tilde{x}_j + B\tilde{u}_j + E\tilde{v}_j, \quad$ with $\tilde{x}_0 = x(k)$
$\tilde{u}_j = F_j \tilde{x}_j + u^0_j, \quad$ for $j \neq 0$ and $\tilde{u}_0 = u^0_0$.

In Problem P1, the weighting matrices in the objective function $\Gamma^x_{j+1} \in \Re^{n_x \times n_x}$ and $\Gamma^u_j \in \Re^{n_v \times n_v}$, $j = 0, 1, \ldots, N-1$, are symmetric and positive definite.

We convert the min–max Problem P1 into a minimization problem by replacing the maximization part in (3) with the following semi-infinite minimization:

$$\min_{\gamma} \gamma, \tag{6}$$

subject to, for all $v_j \in \mathcal{V}_j = \mathcal{E}(Q^v_j, c^v_j),\ j = 0, 1, \ldots, N-1,$

$$\sum_{j=1}^{N} \|\tilde{x}_j\|^2_{\Gamma^x_j} + \sum_{j=0}^{N-1} \|\tilde{u}_j\|^2_{\Gamma^u_j} \le \gamma. \tag{7}$$

For simplicity in this paper, we consider the case without the constraints (5). The proposed approach can be extended to g convex quadratic or linear in x, u, v.

3. Three-Step Procedure

The robust MPC optimization problem is hard to solve directly because the constraints on the feedback gain matrices are nonconvex. We propose the following three-step procedure to approximate Problem P1 with a set of convex optimization problems.

Step 1. Decompose the robust MPC optimization problem into a set of semi-infinite optimizations with one-step look-ahead semi-infinite constraints.

Step 2. Compute ellipsoidal approximations to the sets of reachable states, based on the solution in Step 1 as constraints for Step 3.

Step 3. To reduce conservativeness, optimize the open-loop control sequence in the affine feedback control law over the entire control horizon.

In Step 1, the robust MPC optimization problem is decomposed into the following N single-stage robust MPC optimization problems. For $j = 0, 1, \ldots, N-1$, solve the problem below.

(P2)
$$\min_{F_j, u_j^0, \gamma_{j+1}^x, \gamma_j^u} \gamma_{j+1}^x + \gamma_j^u,$$

subject to, for all $v_j \in \mathcal{E}(Q_j^v, c_j^v)$ and $x_j \in \mathcal{E}(Q_j^x, c_j^x)$, for $j \neq 0$ and $x_0 = x(k)$, $F_0 = 0_{n_u \times n_x}$,

$$\|(A + BF_j)x_j + Bu_j^0 + Ev_j - c_{j+1}^x\|_{Q_{j+1}^x}^2 \leq 1, \tag{8}$$

$$\|(A + BF_j)x_j + Bu_j^0 + Ev_j\|_{\Gamma_{j+1}^x}^2 \leq \gamma_{j+1}^x, \tag{9}$$

$$\|F_j x_j + u_j^0\|_{\Gamma_j^u}^2 \leq \gamma_j^u. \tag{10}$$

The constraint (8) assures that x_{j+1} satisfies the state constraint given by $\mathcal{E}(Q_{j+1}^x, c_{j+1}^x)$; the constraints (9) and (10) minimize the cost term in the objective function for x_{j+1} and u_j respectively. We denote the solution in Step 1 by $\tilde{F}_j$ and $\tilde{u}_j^0$, for $j = 0, 1, \ldots, N-1$.

In Step 2, the following sequence of optimization problems is solved to find ellipsoidal approximations of the sets of reachable states using the control law from Step 1. For $j = 0, 1, \ldots, N-1$, solve the problem below.

(P3)
$$\min_{\tilde{Q}_{j+1}^x, \tilde{c}_{j+1}^x} \det(\tilde{Q}_{j+1}^x)^{-1},$$

subject to, for all $v_j \in \mathcal{E}(Q_j^v, c_j^v)$ and $x_j \in \mathcal{E}(\tilde{Q}_j^x, \tilde{c}_j^x)$, for $j \neq 0$ and $x_0 = x(k)$, $\tilde{F}_0 = 0_{n_u \times n_x}$,

$$\|(A + B\tilde{F}_j)x_j + B\tilde{u}_j^0 + Ev_j - \tilde{c}_{j+1}^x\|_{\tilde{Q}_{j+1}^x}^2 \leq 1, \tag{11}$$

for all $x \in \mathcal{E}(\tilde{Q}_{j+1}^x, \tilde{c}_{j+1}^x)$,

$$\|x - c_{j+1}^x\|_{Q_{j+1}^x}^2 \leq 1. \tag{12}$$

The constraint (11) guarantees that the ellipsoidal approximation contains all possible reachable states starting from the set $\mathcal{E}(\tilde{Q}_j^x, \tilde{c}_j^x)$ or the initial state x_0 under bounded disturbance inputs; the constraint (12) ensures that all points in the ellipsoidal approximation satisfy the state constraints.

The feedback controls from Step 1 can be overlyconservative because they do not coordinate the control over the entire prediction horizon. To compute a less conservative solution, we formulate an optimization problem over the entire prediction horizon in Step 3. The optimization is performed over the open-loop sequence of controls in the affine control law $\mathcal{U}=\{u_0^0, u_1^0, \ldots, u_{N-1}^0\}$ and the centers of the reachable sets of states $\bar{C}=\{\bar{c}_1^x, \bar{c}_2^x, \ldots, \bar{c}_N^x\}$. Precisely, we have the problem below.

(P4) $\min_{\mathcal{U},\bar{C},\gamma} \gamma,$

subject to, for all $v_j \in \mathcal{E}(Q_j^v, c_j^v)$ and for all $x_j \in \mathcal{E}(\tilde{Q}_j^x, \bar{c}_j^x)$, for $j \neq 0$ and $x_0 = x(k)$,

$$\|\tilde{\Phi}_j^{j+1} x_j + Bu_j^0 + Ev_j - \bar{c}_{j+1}^x\|_{\tilde{Q}_{j+1}^x}^2 \leq 1, \quad j=1,2,\ldots,N-1, \tag{13}$$

$$\|x_j - c_j^x\|_{Q_j^x}^2 \leq 1, \quad j=0,1,\ldots,N, \tag{14}$$

$$\sum_{j=1}^{N} \left\| \tilde{\Phi}_0^j x_0 + \sum_{l=0}^{j-1}\left[\tilde{\Phi}_{l+1}^j B u_l^0\right] + \sum_{l=0}^{j-1}\left[\tilde{\Phi}_{l+1}^j E v_l\right] \right\|_{\Gamma_j^x}^2 + \sum_{j=0}^{N-1} \left\| \tilde{F}_j \left(\tilde{\Phi}_0^j x_0 + \sum_{l=0}^{j-1}\left[\tilde{\Phi}_{k+1}^j B u_l^0\right] + \sum_{l=0}^{j-1}\left[\tilde{\Phi}_{k+1}^j E v_l\right] \right) + u_j^0 \right\|_{\Gamma_j^u}^2 \leq \gamma, \tag{15}$$

where the state transition matrix $\tilde{\Phi}_l^j$, $l=0,1,\ldots,j-1$, is given by

$$\tilde{\Phi}_l^j = \prod_{t=l}^{j-1}\left(A + B\tilde{F}_t\right) = \left(A + B\tilde{F}_{j-1}\right)\left(A + B\tilde{F}_{j-2}\right)\cdots\left(A + B\tilde{F}_l\right),$$

with $\tilde{F}_0 = 0_{n_u \times n_x}$. (16)

The constraint (13) requires that the ellipsoids with new centers still contain the sets of reachable states. The constraint (14) plays the same role as the constraint (12) in Problem P3 for the reachable set approximation. The constraint (15) is a restatement of the constraint (7).

The following proposition establishes the correctness of the three-step procedure.

Proposition 3.1. If the constraints in the three-step procedure are feasible, the three-step procedure provides a feasible solution to the robust MPC optimization Problem P1.

Proof. This is straightforward, because the constraint (15) is a reformulation of (7) and because the constraints (14) for $j=1,2,\ldots,N$ imply (4). □

4. Three-Step Procedure Using LMI

We apply the $\mathcal{S}$-procedure (see Ref. 15, pp. 23) to obtain sufficient LMI constraints to replace the semi-infinite constraints in the optimizations Problems P2–P4. The $\mathcal{S}$-procedure for multiple quadratic forms says that, for a given set of matrices $P_0,\ldots,P_n$, a sufficient condition for

$$z^T P_1 z \le 0, z^T P_2 z \le 0, \ldots, z^T P_n z \le 0, \Rightarrow z^T P_0 z \le 0, \quad \text{for all}\, z,$$

is that there exists nonnegative numbers $\tau_1,\tau_2,\ldots,\tau_n$ such that

$$z^T P_0 z - \sum_{i=1}^{n} \tau_i z^T P_i z \le 0, \quad \text{for all } z;$$

that is,

$$P_0 - \sum_{i=1}^{n} \tau_i P_i \preceq 0,$$

where $P \preceq 0$ means that P is negative semidefinite.

4.1. LMI Optimization for Single-Stage Robust MPC.

Following the S-procedure and the Schur complement computation, we have the following LMI constraints for the semi-infinite constraint (8)–(10): for $j=0$,

$$\begin{bmatrix} -(Q_1^x)^{-1} & E & (Ax_0+Bu_0^0-c_1^x) \\ E^T & -t_0^v Q_0^v & t_0^v Q_0^v c_0^v \\ (Ax_0+Bu_0^0-c_1^x)^T & t_0^v (C_0^v) Q_0^v & -t_0^v\left[(C_0^v)^T Q_0^v c_0^v - 1\right]-1 \end{bmatrix} \preceq 0, \quad (17)$$

$$\begin{bmatrix} -(\Gamma_1^x)^{-1} & E & (Ax_0+Bu_0^0) \\ E^T & -t_0^{J,v} Q_0^v & t_0^{J,v} Q_0^v c_0^v \\ (Ax_0+Bu_0^0)^T & t_0^{J,v} (C_0^v)^T Q_0^v & -t_0^{J,v}\left[(C_0^v)^T Q_0^v c_0^v - 1\right]-\gamma_1^x \end{bmatrix} \preceq 0, \quad (18)$$

$$\begin{bmatrix} -(\Gamma_0^u)^{-1} & u_0^0 \\ (u_0^0)^T & -\gamma_0^u \end{bmatrix} \preceq 0, \quad (19)$$

and for $j = 1, 2, \ldots, N-1$,

$$\begin{bmatrix} -(Q^x_{j+1})^{-1} & A+BF_j & E & Bu^0_j - c^x_{j+1} \\ (A+BF_j)^T & -t^x_j Q^x_j & 0_{n_x \times n_v} & t^x_j Q^x_j c^x_j \\ E^T & 0_{n_v \times n_x} & -t^v_j Q^v_j & t^v_j Q^v_j c^v_j \\ (Bu^0_j - c^x_{j+1})^T & t^x_j \left(c^x_j\right)^T Q^x_j & t^v_j \left(c^v_j\right)^T Q^v_j & \begin{matrix} -t^x_j\left[(c^x_j)^T Q^x_j c^x_j - 1\right] \\ -t^v_j\left[(c^v_j)^T Q^v_j c^v_j - 1\right] - 1 \end{matrix} \end{bmatrix} \preceq 0, \quad (20)$$

$$\begin{bmatrix} -(\Gamma^x_{j+1})^{-1} & A+BF_j & E & Bu^0_j \\ (A+BF_j)^T & -t^{J,x}_j Q^x_j & 0_{n_x \times n_v} & t^{J,x}_j Q^x_j c^x_j \\ E^T & 0_{n_v \times n_x} & -t^{J,v}_j Q^v_j & t^{J,v}_j Q^v_j c^v_j \\ (Bu^0_j)^T & t^{J,x}_j \left(c^x_j\right)^T Q^x_j & t^{J,v}_j \left(c^v_j\right)^T Q^v_j & \begin{matrix} -t^{J,x}_j\left[(c^x_j)^T Q^x_j c^x_j - 1\right] \\ -t^{J,v}_j\left[(c^v_j)^T Q^v_j c^v_j - 1\right] \\ -\gamma^x_j \end{matrix} \end{bmatrix} \preceq 0, \quad (21)$$

$$\begin{bmatrix} -(\Gamma^u_j)^{-1} & F_j & u^0_j \\ F_j^T & -t^{J,u}_j Q^x_j & t^{J,u}_j Q^x_j c^x_j \\ \left(u^0_j\right)^T & t^{J,u}_j \left(c^x_j\right)^T Q^x_j & -t^{J,u}_j\left[(c^x_j)^T Q^x_j c^x_j - 1\right] - \gamma^u_j \end{bmatrix} \preceq 0. \quad (22)$$

We have the following optimizations with LMI constraints corresponding to P2:

(P5(a)) $\min_{u^0_0, t^v_0, t^{J,v}_0, \gamma^x_1, \gamma^u_0} \gamma^x_1 + \gamma^u_0$,
subject to (18), (19), (17), with $x_0 = x(k)$,
and for $j = 1, 2, \ldots, N-1$, solve the problem below.

(P5(b)) $\min_{F_j, u^0_j, t^x_j, t^v_j, t^{J,x}_j, t^{J,v}_j, t^{J,u}_j, \gamma^x_{j+1}, \gamma^u_j} \gamma^x_{j+1} + \gamma^u_j$, subject to (21), (22), (20).

We denote the solutions for Problems P5(a) and P5(b) by

$$\left\{\tilde{F}_j\right\}_{j=1}^{N-1}, \left\{\tilde{u}^0_j\right\}_{j=0}^{N-1}, \left\{\tilde{t}^x_j\right\}_{j=1}^{N-1}, \left\{\tilde{t}^v_j\right\}_{j=0}^{N-1}, \left\{\tilde{t}^{J,x}_j\right\}_{j=1}^{N-1}, \left\{\tilde{t}^{J,u}_j\right\}_{j=1}^{N-1},$$
$$\left\{\tilde{t}^{J,v}_j\right\}_{j=0}^{N-1}, \left\{\tilde{\gamma}^x_j\right\}_{j=1}^{N}, \left\{\tilde{\gamma}^u_j\right\}_{j=0}^{N-1}.$$

4.2. Ellipsoidal Approximation for the Reachable States. Similarly, we have the following LMIs corresponding to the constraints (12) and (11):

$$\begin{bmatrix} Q_j^x & -Q_j^x c_j^x \\ -\left(c_j^x\right)^T Q_j^x & \left(c_j^x\right)^T Q_j^x c_j^x - 1 \end{bmatrix} - \begin{bmatrix} \hat{Q}_j^x & \hat{b}_j^x \\ \left(\hat{b}_j^x\right)^T & \hat{e}_j^x \end{bmatrix} \preceq 0, \quad j = 1, 2, \ldots, N, \tag{23}$$

$$\begin{bmatrix} E^T \hat{Q}_1^x E & E^T \left(\hat{Q}_1^x \hat{a}_1^x + \hat{b}_1^x\right) \\ \left(\hat{Q}_1 \hat{a}_1 + \hat{b}_1^x\right)^T E & \hat{a}_1^T \hat{Q}_1^x \hat{a}_1 + 2\hat{a}_1^T \hat{b}_1^x + \hat{e}_1^x \end{bmatrix} - \begin{bmatrix} \hat{t}_0^v Q_0^v & -\hat{t}_0^v Q_0^v c_0^v \\ -\hat{t}_0^v \left(c_0^v\right)^T Q_0^v & \hat{t}_0^v \left((c_0^v)^T Q_0^v c_0^v - 1\right) \end{bmatrix} \preceq 0, \tag{24}$$

where

$$\hat{a}_1 = \left(A + B\tilde{F}_0\right) x_0 + B\tilde{u}_0^0$$

and, for $j = 1, 2, \ldots, N-1$,

$$\begin{bmatrix} \left(A+B\tilde{F}_j\right)^T \hat{Q}_{j+1}^x \left(A+B\tilde{F}_j\right) & \left(A+B\tilde{F}_j\right)^T \hat{Q}_{j+1}^x E & \left(A+B\tilde{F}_j\right)^T \left(\hat{Q}_{j+1}^x B\tilde{u}_j^0 + \hat{b}_{j+1}^x\right) \\ E^T \hat{Q}_{j+1}^x \left(A+B\tilde{F}_j\right) & E^T \hat{Q}_{j+1}^x E^T & E^T \left(\hat{Q}_{j+1}^x B\tilde{u}_j^0 + \hat{b}_{j+1}^x\right) \\ \left(\hat{Q}_{j+1}^x B\tilde{u}_j^0 + \hat{b}_{j+1}^x\right)^T \left(A+B\tilde{F}_j\right) & \left(\hat{Q}_{j+1}^x B\tilde{u}_j^0 + \hat{b}_{j+1}^x\right)^T E & (B\tilde{u}_j^0)^T \hat{Q}_{j+1}^x B\tilde{u}_j^0 + 2(B\tilde{u}_j^0)^T \hat{b}_{j+1}^x + \hat{e}_{j+1}^x \end{bmatrix}$$

$$\begin{bmatrix} \hat{t}_j^x \tilde{Q}_j^x & 0_{n_x \times n_v} & -\hat{t}_j^x \tilde{Q}_j^x \tilde{c}_j^x \\ 0_{n_v \times n_x} & \hat{t}_j^v Q_j^v & -\hat{t}_j^v Q_j^v c_j^v \\ -\hat{t}_j^x (\tilde{c}_j^x)^T \tilde{Q}_j^x & -\hat{t}_j^v (c_j^v)^T & Q_j^v \hat{t}_j^x \left((\tilde{c}_j^x)^T \tilde{Q}_j^x \tilde{c}_j^x - 1\right) + \hat{t}_j^v \left((c_j^v)^T Q_j^v c_j^v - 1\right) \end{bmatrix} \preceq 0. \tag{25}$$

The optimizations Problem P3 can be converted to max-det optimizations:

(P6(a)) $\min_{\hat{Q}_1^x, \hat{b}_1^x, \hat{e}_1^x, \hat{t}_1^v} \log\left(\det\left(\left(\hat{Q}_1^x\right)^{-1}\right)\right)$,
subject to (24) and (23) with $x_0 = x(k)$ and
for $j = 1, 2, \ldots, N-1$, solve the problem below.

(P6(b)) $\min_{\hat{Q}_{j+1}^x, \hat{b}_{j+1}^x, \hat{e}_{j+1}^x, \hat{t}_j^x, \hat{t}_j^v} \log\left(\det\left(\left(\hat{Q}_{j+1}^x\right)^{-1}\right)\right)$,
subject to (25) and (23).

Note that $\tilde{Q}_j^x$ and $\tilde{c}_j^x$ for the ellipsoidal approximation $\mathcal{E}\left(\tilde{Q}_j^x, \tilde{c}_j^x\right)$ are computed for $j = 1, 2, \ldots, N$ by

$$\hat{Q}_j^x = \hat{Q}_j^x \Big/ \left[\left(\hat{b}_j^x\right)^T \left(\hat{Q}_j^x\right)^{-1} \hat{b}_j^x - \hat{e}_j^x \right] \quad \text{and} \quad \tilde{c}_j^x = -\left(\hat{Q}_j^x\right)^{-1} \hat{b}_j^x.$$

4.3. LMI Optimization for Full-Horizon Robust MPC. By the same process, we have the following sufficient LMI constraints for (13),

$$\begin{bmatrix} \left(\tilde{Q}_{j+1}^x\right) & A+B\tilde{F}_j & E & \left(A+B\tilde{F}_j\right)\bar{c}_j^x + Bu_j^0 - \bar{c}_{j+1}^x \\ \left(A+B\tilde{F}_j\right)^T & -t_j^x \tilde{Q}_j^x & 0_{n_x\times n_v} & 0_{n_x\times 1} \\ E^T & 0_{n_v\times n_x} & -t_j^v Q_j^v & t_j^v Q_j^v c_j^v \\ \left(\left(A+B\tilde{F}_j\right)\bar{c}_j^x + Bu_j^0 - \bar{c}_{j+1}^x\right)^T & 0_{1\times n_x} & t_j^v \left(c_j^v\right)^T Q_j^v & t_j^x - t_j^v\left(\left(c_j^v\right)^T Q_j^v c_j^v - 1\right) - 1 \end{bmatrix} \preceq 0, \tag{26}$$

for $j = 1, 2, \ldots, N-1$, and

$$\begin{bmatrix} \left(\tilde{Q}_1^x\right)^{-1} & E & Ax_0 + Bu_0^0 - \bar{c}_1^x \\ E^T & -t_0^v Q_0^v & t_0^v Q_0^v c_0^v \\ (Ax_0 + Bu_0^0 - \bar{c}_1^x)^T & t_0^v (c_0^v)^T Q_0^v & -t_0^v\left((c_0^v)^T Q_0^v c_0^v - 1\right) - 1 \end{bmatrix} \preceq 0, \tag{27}$$

and a sufficient LMI constraint for (14),

$$\begin{bmatrix} -(P_j^x)^{-1} & I^{n_x\times n_x} & -(c_j^x - \bar{c}_j^x) \\ I^{n_x\times n_x} & -\bar{\tau}_j \tilde{Q}_j^x & 0_{n_x\times 1} \\ -(c_j^x - \bar{c}_j^x)^T & 0_{1\times n_x} & \bar{\tau}_j - 1 \end{bmatrix} \preceq 0, \tag{28}$$

with $\bar{\tau}_j \geq 0$ for $j = 1, 2, \ldots, N$.

For the constraints (15), we apply the following decomposition:

$$\gamma = \sum_{j=0}^{N} \gamma_j^x + \sum_{j=0}^{N-1} \gamma_j^u, \tag{29}$$

$$\gamma_j^x \geq \left\| \tilde{\Phi}_0^j x_0 + \sum_{l=0}^{j-1}\left[\tilde{\Phi}_{l+1}^j B u_l^0\right] + \sum_{l=0}^{j-1}\left[\tilde{\Phi}_{l+1}^j E v_l\right] \right\|_{\Gamma_j^x}^2, \quad j = 1, 2, \ldots, N, \tag{30}$$

$$\gamma_j^u \geq \left\| \tilde{F}_j \tilde{\Phi}_0^j x_0 + \sum_{l=0}^{j-1}\left[\tilde{F}_j \tilde{\Phi}_{k+1}^j B u_l^0\right] + \sum_{l=0}^{j-1}\left[\tilde{F}_j \tilde{\Phi}_{l+1}^j E v_l\right] + u_j^0 \right\|_{\Gamma_j^u}^2,$$

$$j = 1, 2, \ldots, N-1, \tag{31}$$

$$\gamma_0^u \geq \left\| u_0^0 \right\|_{\Gamma_0^u}^2. \tag{32}$$

We have the following LMI constraints corresponding to (30)–(32):

$$\begin{bmatrix} -\left(\Gamma_j^x\right)^{-1} & \tilde{\mathbb{G}}_j & \tilde{a}_j^J \\ \tilde{\mathbb{G}}_j^T & -\mathbb{Q}_{j,1}^v(T_j^{J,x}) & -\mathbb{Q}_{j,2}^v(T_j^{J,x}) \\ \left(\tilde{a}_j^J\right)^T & -\left(\mathbb{Q}_{j,2}^v(T_j^{J,x})\right)^T & -\mathbb{Q}_{j,3}^v(T_j^{J,x}) - \gamma_j^x \end{bmatrix} \preceq 0, \quad j = 1, 2, \ldots, N, \tag{33}$$

$$\begin{bmatrix} -(\Gamma_j^u)^{-1} & \tilde{F}_j \tilde{\mathbb{G}}_j & \tilde{F}_j \tilde{a}_j^J + u_j^0 \\ \left(\tilde{F}_j \tilde{\mathbb{G}}_j\right)^T & -\mathbb{Q}_{j,1}^v\left(T_j^{J,u}\right) & -\mathbb{Q}_{j,2}^v\left(T_j^{J,u}\right) \\ \left(\tilde{F}_j \tilde{a}_j^J + u_j^0\right)^T & -\left(\mathbb{Q}_{j,2}^v\left(T_j^{J,u}\right)\right)^T & -\mathbb{Q}_{j,3}^v\left(T_j^{J,u}\right) - \gamma_j^u \end{bmatrix} \preceq 0,$$
$$j = 1, 2, \ldots, N-1, \tag{34}$$

$$\begin{bmatrix} -(\Gamma_0^u)^{-1} & \tilde{F}_0 x_0 + u_0^0 \\ \left(\tilde{F}_0 x_0 + u_0^0\right)^T & -\gamma_0^u \end{bmatrix} \preceq 0, \tag{35}$$

where

$$\tilde{\mathbb{G}}_j = \left[\tilde{G}_{0,j}^T \tilde{G}_{1,j}^T \ldots \tilde{G}_{j-1,j}^T\right]^T, \quad \tilde{G}_{l,j} = \tilde{\Phi}_{l+1}^j E, \quad l = 0, 1, \ldots, j-1,$$

$$\tilde{a}_j^J = \tilde{\Phi}_0^j x_0 + \sum_{l=0}^{j-1} \left[\tilde{\Phi}_{l+1}^j B u_l^0\right], \quad T_j^{J,x} = \left\{t_{0,j}^{J,x}, t_{1,j}^{J,x}, \ldots, t_{j-1,j}^{J,x}\right\}, \quad j = 1, 2, \ldots, N,$$

$$T_j^{J,u} = \left\{t_{0,j}^{J,u}, t_{1,j}^{J,u}, \ldots, t_{j-1,j}^{J,u}\right\}, \quad j = 1, 2, \ldots, N-1.$$

$\mathbb{Q}_{j,1}^v(\cdot)$, $\mathbb{Q}_{j,2}^v(\cdot)$, $\mathbb{Q}_{j,3}^v(\cdot)$ are given by

$$\mathbb{Q}_{j,1}^v(T_j) = \begin{bmatrix} t_{0,j} Q_0^v & & \\ & \ddots & \\ & & t_{j-1,j} Q_{j-1}^v \end{bmatrix}, \quad \mathbb{Q}_{j,2}^v(T_j) = \begin{bmatrix} -t_{0,j} Q_0^v c_0^v \\ \vdots \\ -t_{j-1,j} Q_{j-1}^v c_{j-1}^v \end{bmatrix},$$

$$\mathbb{Q}_{j,3}^v(T_j) = \sum_{k=0}^{j-1} t_{k,j} \left[\|c_k^v\|_{Q_k^v}^2 - 1\right].$$

The LMI optimization for the full-horizon robust MPC optimization P4 is

(P7) $\quad \min_{\mathcal{U}, \bar{C}, T, \Upsilon} \sum_{j=1}^{N} \gamma_j^x + \sum_{j=0}^{N-1} \gamma_j^u,$
with $x_0 = x(k)$,
subject to (26)–(28) and (33)–(35)

where

$$\mathcal{U}=\{u_0^0, u_1^0, \ldots, u_{N-1}^0\},\ \ \bar{C}=\{\bar{c}_1^x, \bar{c}_2^x, \ldots, \bar{c}_N^x\},$$

$$T=\left\{\left\{T_j^{J,x}\right\}_{j=1}^{N},\ \left\{T_j^{J,u}\right\}_{j=1}^{N-1},\ \{t_j^x\}_{j=1}^{N-1},\ \{t_j^v\}_{j=0}^{N-1},\ \{\bar{\tau}_j\}_{j=1}^{N}\right\},$$

$$\Upsilon=\left\{\left\{\gamma_j^x\right\}_1^N,\ \left\{\gamma_j^u\right\}_0^{N-1}\right\}.$$

5. Feasibility and Closed-Loop Stability

In our approach, feasibility and stability issues are addressed by propagating the reachable state constraints forward at each control step and by imposing a control invariant set as the terminal constraint.

Definition 5.1. Control-Invariant Set (see Ref. 16). Given a disturbance signal set $v \in \mathcal{V}$, a set $\mathcal{X}_\mathcal{I} \subseteq \Re^{n_x}$ is control invariant if there exists a pair of $F_\mathcal{I}$ and $u_\mathcal{I}^0$ such that $(A+BF_\mathcal{I})x + Bu_\mathcal{I}^0 + Ev \in \mathcal{X}_\mathcal{I}$, for all $x \in \mathcal{X}_\mathcal{I}, v \in \mathcal{V}$.

The controller uses the predictions of the reachable states at the previous control step to update the ellipsoidal state constraints

$$\begin{aligned} &Q_j^x = \tilde{Q}_{j+1,k-1}, c_j^x = \bar{c}^x_{j+1,k-1}, \quad \text{for } j=1,2,\ldots,N-1,\\ &Q_N^x = Q_\mathcal{I}^x, c_N^x = c_\mathcal{I}^x, \end{aligned} \tag{36}$$

where $\mathcal{E}(Q_\mathcal{I}^x, c_\mathcal{I}^x)$ is an ellipsoidal control invariant set with control parameters $F_\mathcal{I}$ and $u_\mathcal{I}^0$. The subscripts $k-1$ label the values of the corresponding parameters or variables at time step $k-1$. We assume that the updating of the disturbance bounds satisfies

$$\mathcal{V}_j \subseteq \mathcal{V}_{j+1,k+1}, i=0,1,\ldots,N-2, \text{ and } \mathcal{V}_{N-1} \subseteq \mathcal{V}, \tag{37}$$

where $\mathcal{V}$ is the bound on the disturbance corresponding to the control invariant set $\mathcal{E}(Q_\mathcal{I}^x, c_\mathcal{I}^x)$. We require $\mathcal{E}(Q_\mathcal{I}^x, c_\mathcal{I}^x)$ to satisfy the following sufficient LMI condition:

$$\begin{bmatrix} -(Q_\mathcal{I}^x)^{-1} & A+BF_\mathcal{I} & E & Bu_\mathcal{I}^0 - c_\mathcal{I}^x \\ (A+BF_\mathcal{I})^T & -t_\mathcal{I}^x Q_\mathcal{I}^x & 0_{n_x \times n_v} & t_\mathcal{I}^x Q_\mathcal{I}^x c_\mathcal{I}^x \\ E^T & 0_{n_v \times n_x} & -t_\mathcal{I}^v Q^v & t_\mathcal{I}^v Q^v c^v \\ (Bu_\mathcal{I}^0 - c_\mathcal{I}^x)^T & t_\mathcal{I}^x (c_\mathcal{I}^x)^T Q_\mathcal{I}^x & t_\mathcal{I}^v (c^v)^T Q^v & \begin{array}{l} -t_\mathcal{I}^x\left((c_\mathcal{I}^x)^T Q_\mathcal{I}^x c_\mathcal{I}^x - 1\right) \\ -t_\mathcal{I}^v\left((c^v)^T Q^v c^v - 1\right) - 1 \end{array} \end{bmatrix} \preceq 0, \tag{38}$$

Theorem 5.1. If the optimization problems P5a and P5b for $j = 1, 2, \dots, N-1$ are feasible at the initial control step $k=0$, then the optimization problems P5a, P5b, P6a, P6b, P7 are feasible at all control steps $k \geq 0$. Furthermore, the state of the system goes into the control invariant set in N steps.

To prove Theorem 5.1, we using the following lemmas from Ref. 17.

Lemma 5.1. If the single-stage optimization problems P5a and P5b for $j=1, 2, \dots, N-1$ are feasible with $Q_N^x = Q_{\mathcal{I}}^x$ and $c_N^x = c_{\mathcal{I}}^x$, the optimization problems P6a and P6b for $j=1, 2, \dots, N-1$ for ellipsoidal approximations of the reachable states are feasible.

Lemma 5.2. If the optimization problems P6a and P6b for $j = 1, 2, \dots, N-1$ are feasible, the full-horizon optimization problem P7 is feasible.

Lemma 5.3. Assume that there exist two nonnegative variables t_x and t_v such that

$$\begin{bmatrix} -(Q_1)^{-1} & A+BF & E & B_u - c_1 \\ A^T + F^T B^T & -t_x Q_0 & 0_{n_x \times n_v} & t_x Q_0 c_0 \\ E^T & 0_{n_v \times n_x} & -t_v Q_v & t_v Q_v c_v \\ u^T B^T - c_1^T & t_x c_0^T Q_0 & t_v c_v^T Q^v & -t_x(c_0^T Q_0 c_0 + 1) \\ & & & -t_v(c_v^T Q_v c_v - 1) - 1 \end{bmatrix} \preceq 0, \quad (39)$$

where $Q_1 \in \Re^{n_x \times n_x}$, $Q_0 \in \Re^{n_x \times n_x}$, $Q_v \in \Re^{n_v \times n_v}$ are symmetric positive-definite matrices and $c_1 \in \Re^{n_x}$, $c_0 \in \Re^{n_x}$, $c_v \in \Re^{n_v}$, $A \in \Re^{n_x \times n_x}$, $B \in \Re^{n_x \times n_u}$, $E \in \Re^{n_x \times n_v}$, $F \in \Re^{n_u \times n_x}$, $u \in \Re^{n_u}$. Then, for arbitrary symmetric positive matrix $\Gamma \in \Re^{n_x \times n_x}$, there exist three nonnegative numbers γ, t_x^J, t_v^J such that

$$\begin{bmatrix} -(\Gamma)^{-1} & A+BF & E & Bu \\ A^T + F^T B^T & -t_x^J Q_0 & 0_{n_x \times nv} & t_x^J Q_0 c_0 \\ E^T & 0_{n_v \times n_x} & -t_v^J Q_v & t_v^J Q_v c_v \\ u^T B^T & t_x^J c_0^T Q_0 & t_v^J c_v^T Q^v & -t_x^J (c_0^T Q_0 c_0 + 1) - t_v^J (c_v^T Q_v c_v - 1) - \gamma \end{bmatrix} \preceq 0. \quad (40)$$

Proof of Theorem 5.1. We prove this theorem by induction. At the initial control step $k=0$, the single-stage LMI optimization problems P5(a) and P5(b) for $j=1, 2, \dots, N-1$ are feasible.

Suppose that at the control step $k = K$, the optimization problems P5(a) and P5(b) for $j = 1, 2, \ldots, N-1$ are feasible. Denote the optimal solutions by

$$\tilde{\mathcal{F}}_K = \left\{ \tilde{F}_{j,K} \right\}_{j=1}^{N-1}, \quad \tilde{\mathcal{U}}_K = \left\{ \tilde{u}^0_{j,K} \right\}_{j=0}^{N-1},$$

$$\tilde{T}_K = \left\{ \left\{ \tilde{t}^x_{j,K} \right\}_{j=1}^{N-1}, \left\{ \tilde{t}^v_{j,K} \right\}_{j=0}^{N-1}, \left\{ \tilde{t}^{J,x}_{j,K} \right\}_{j=1}^{N-1}, \left\{ \tilde{t}^{J,v}_{j,K} \right\}_{j=0}^{N-1}, \left\{ \tilde{t}^{J,u}_{j,K} \right\}_{j=1}^{N-1} \right\},$$

$$\tilde{\Upsilon}_K = \left\{ \left\{ \tilde{\gamma}^x_{j,K} \right\}_{j=1}^{N}, \left\{ \tilde{\gamma}^u_{j,K} \right\}_{j=0}^{N-1} \right\}.$$

From Lemmas 5.1 and 5.2, the optimization problems P6a, P6b for $j = 1, 2, \ldots, N-1$ and P7 are feasible. Denote the optimal solution for Problems P6a and P6b by

$$\tilde{\mathcal{Q}}_K = \left\{ \tilde{\mathcal{Q}}^x_{j,K} \right\}_{j=1}^{N}, \quad \tilde{\mathcal{C}}^x_K = \left\{ \tilde{c}^x_{j,K} \right\}_{j=1}^{N}$$

and the optimal solution for Problem P7 by

$$\begin{aligned}
\bar{u}_K &= \left\{ \bar{u}^0_{j,0} \right\}_{j=1}^{N-1}, \quad \bar{C}_K = \left\{ \bar{c}^x_j \right\}_{j=1}^{N}, \\
\bar{T}_K &= \left\{ \left\{ \bar{T}^{j,x}_{j,K} \right\}_{j=1}^{N}, \quad \left\{ \bar{T}^{J,u}_{j,K} \right\}_{j=1}^{N-1}, \left\{ \bar{t}^x_{j,K} \right\}_{j=1}^{N-1}, \left\{ \bar{t}^v_{j,K} \right\}_{j=0}^{N-1}, \left\{ \bar{\tau}_{j,K} \right\}_{j=1}^{N} \right\}, \\
\bar{\Upsilon} &= \left\{ \left\{ \bar{\gamma}^x_{j,K} \right\}_{j=1}^{N}, \quad \left\{ \bar{\gamma}^u_{j,K} \right\}_{j=1}^{N-1} \right\}.
\end{aligned}$$

From the LMI constraints (26) for $j = 1$, we have

$$\begin{bmatrix}
-\left(\tilde{Q}^x_{2,K}\right)^{-1} & A + B\tilde{F}_{1,K} & E & B\bar{u}^0_{1,K} - \bar{c}^x_{2,K} \\
\left(A + B\tilde{F}_{1,K}\right)^T & -\hat{t}^x_{1,K}\tilde{Q}^x_{1,K} & 0_{n_x \times nv} & \hat{t}^x_{1,K}\tilde{Q}^x_{1,K}\bar{c}^x_{1,K} \\
E^T & 0_{n_v \times n_x} & -\hat{t}^v_{1,K}Q^v_{1,K} & \hat{t}^v_{1,K}Q^v_{1,K}c^v_{1,K} \\
\left(B\bar{u}^0_{1,K} - \bar{c}^x_{2,K}\right)^T & \hat{t}^x_{1,K}\left(\bar{c}^x_{1,K}\right)^T\tilde{Q}^x_{1,K} & \hat{t}^v_{1,K}\left(c^v_{1,K}\right)^T Q^v_{1,K} & \begin{array}{l} -\hat{t}^x_{1,K}\left(\left(\bar{c}^x_{1,K}\right)^T\tilde{Q}^x_{1,K}\bar{c}^x_{1,K} - 1\right) \\ -\hat{t}^v_{1,K}\left(\left(c^v_{1,K}\right)^T Q^v_{1,K}c^v_{1,K} - 1\right) - 1 \end{array}
\end{bmatrix} \preceq 0. \tag{41}$$

Since $x(K+1)$ is inside the ellipsoid $\mathcal{E}\left(\tilde{Q}^x_{1,K}, \bar{c}^x_{1,K}\right)$, there exists an ellipsoid centered at $x(K+1)$ that is contained in it. This implies that

$$\begin{bmatrix} \tilde{Q}^x_{1,K} & -\tilde{Q}^x_{1,K}\bar{c}^x_{1,K} \\ -\left(\bar{c}^x_{1,K}\right)^T \tilde{Q}^x_{1,K} & \left(\bar{c}^x_{1,K}\right)^T \tilde{Q}_{1,K}\bar{c}^x_{1,K}-1 \end{bmatrix}$$
$$-\tilde{\tau}\begin{bmatrix} Q & -Qx(K+1) \\ -x^T(K+1)Q & x^T(K+1)Qx(K+1)-1 \end{bmatrix} \preceq 0. \qquad (42)$$

According to the uncertainty bound information updating process (37), we have

$$\begin{bmatrix} Q^v_{1,K} & Q^v_{1,K}c^v_{1,K} \\ \left(c^v_{1,K}\right)^T Q^v_{1,K} & \left(c^v_{1,K}\right)^T Q^v_{1,K}c^v_{1,K}-1 \end{bmatrix}$$
$$-\tau^v_0\begin{bmatrix} Q^v_0 & Q^v_0 c^v_0 \\ \left(c^v_0\right)^T Q^v_0 & \left(c^v_0\right)^T Q^v_0 c^v_0-1 \end{bmatrix} \preceq 0. \qquad (43)$$

With the state constraint updating rule (36), we have

$$\begin{bmatrix} -\left(Q^x_1\right)^{-1} & E & Ax_0+Bu^0_0-c^x_1 \\ E^T & -t^v_0 Q^v_0 & t^v_{1,0}Q^v_0 c^v_0 \\ \left(Ax_0+Bu^0_0-c^x_1\right)^T & t^v_0\left(c^v_0\right)^T Q^v_0 & -t^v_0\left(\left(c^v_0\right)^T Q^v_0 c^v_0-1\right)-1 \end{bmatrix} \preceq 0,$$

by combining (41)–(43), where

$$x_0 = x(K+1), \quad t^v_0 = \bar{t}^v_{1,K}\tau^v_0.$$

The LMI constraints (26) for $j=2,\dots,N-1$, combined with the uncertainty bound information updating process (37) and the state constraint updating rule (36), imply the feasibility of (20) for $j=1,\dots,N-2$. The LMI condition (38) guarantees the feasibility of the LMI constraint (17) for $j=N-1$. Lemma 5.3 implies the feasibility of (18)–(22).

Therefore, all the optimization problems with LMI constraints are feasible at all control steps $k \geq 0$ and it follows from the results in Ref. 10 that the state of the system goes to the control invariant set in N steps. □

Algorithms for computing ellipsoidal control invariant sets for systems described by a polytopic set of linear models without disturbances are given in Refs. 6, 18. Schemes to search control invariant sets for systems with bounded disturbances are proposed in Refs. 17, 19.

6. Example

We consider the solenoid system in Figure 1 modeled by

$$\begin{bmatrix} x^1(k+1) \\ x^2(k+1) \end{bmatrix} = \begin{bmatrix} 0.6148 & 0.0315 \\ -0.3155 & -0.0162 \end{bmatrix} \begin{bmatrix} x^1(k) \\ x^2(k) \end{bmatrix} + \begin{bmatrix} 0.0385 \\ 0.0315 \end{bmatrix} u(k) + \begin{bmatrix} 0.0385 \\ 0.0315 \end{bmatrix} v(k),$$

where x^1 and x^2 are the position and the velocity of the plate, the magnetic force u is the control variable, and v is the external disturbance to the system, which is bounded in the range $[-1, 1]$. The parameters of the ellipsoidal state constraint are given as $Q^x = I_{2\times2}$ and $c^x = 0_{2\times1}$.

As the system is already stable, the objective is to apply a magnetic force as little as possible to satisfy the state constraints. The weighting

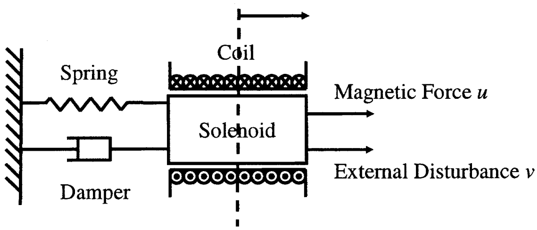

Fig. 1. Solenoid system example.

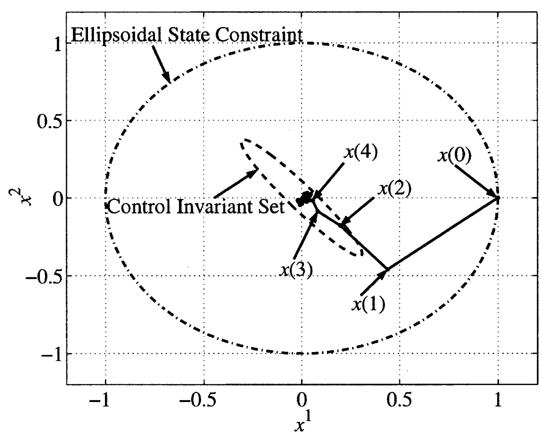

Fig. 2. Simulation results: System response.

matrices in the cost function are $\Gamma^x = I_{2\times 2}$ and $\Gamma^u = 100$. An ellipsoidal control invariant set $\mathcal{E}(Q_{\mathcal{I}}, c_{\mathcal{I}})$ is given by

$$Q_{\mathcal{I}} = \begin{bmatrix} 122.9951 & 97.1845 \\ 97.1845 & 83.9281 \end{bmatrix}, \quad c_{\mathcal{I}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with the corresponding control parameters

$$F_{\mathcal{I}} = [-5.5377 - 0.2842] \quad \text{and} \quad u^0_{\mathcal{I}} = 0.$$

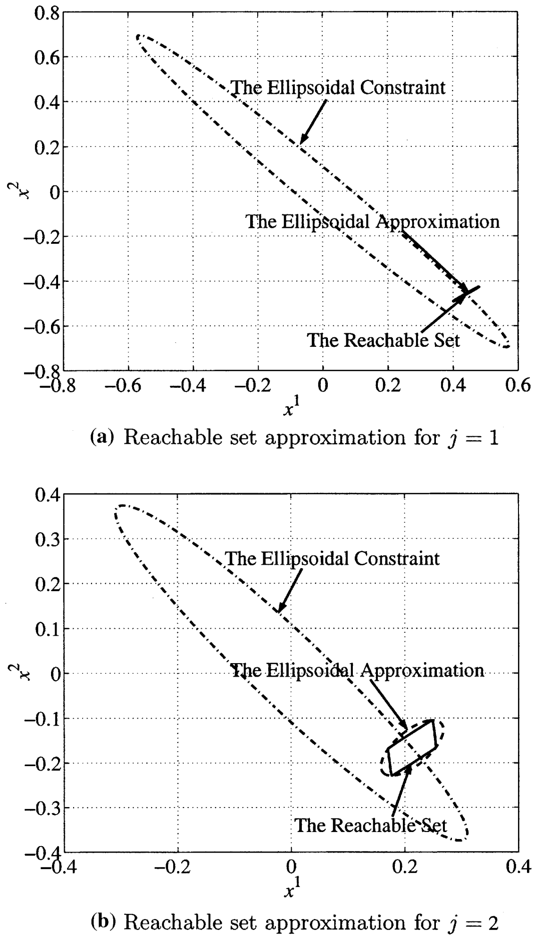

Fig. 3. Simulation results: Reachable sets computed at $k = 0$ by min-max MPC without robustness constraint.

The system response obtained from the three-step robust MPC control procedure is shown in Figure 2, where the initial state of the system is $[1\ 0]^T$ and the prediction horizon is $N=2$. As predicted by Theorem 5.1, the state of the system goes into the control invariant set within 2 steps. In this example, the system converges to a much smaller invariant set than the control invariant set. Figure 3 illustrates the need for the robustness constraint: although the nominal values of the state satisfy the state constraints, there exists a disturbance sequence that can cause constraint violation if the roubustness constraints are not applied.

7. Discussion

This paper proposes a three-step procedure for the robust MPC optimization problem for LTI systems using convex optimizations with LMI constraints. Existing numerical routines can be used to solve the optimization problems, the sizes of which are polynomial in the number of state variables, the number of control variables, and the prediction horizon. Our method does not require open-loop stability.

The complexity of the computations and the conservativeness of the approximations are issues for further investigation. Both these issues need to be addressed for LMI methods in general. Our experience with larger examples (seven to ten state variables) shows that the approach can be applied to systems with sampling times of a fewminutes using standard PCs and existing LMI software packages without any customization to optimize performance (Ref. 17). This is reasonable for some process control applications, but methods for reducing the sizes of the LMI routines would broaden the appeal of the approach for online control.

References

1. Camacho, E. F., and Bordons, C., *Model Predictive Control in the Process Industry*, Advances in Industrial Control, Springer Verlag, New York, NY, 1995.
2. Morari, M., and Lee, J. H., *Model Predictive Control: Past, Present, and Future*, Computers and Chemical Engineering, Vol. 23, pp. 667–682, 1999.
3. Qin, S. J., and Badgwell, T. A., *An Overview of Nonlinear Model Predictive Control Applications*, Nonlinear Model Predictive Control, Edited by F. Allgwer and A. Zhang, Birkhauser, Basel, Switzerland, pp. 369–393, 2000.
4. Rawlings, J. B., *Tutorial Overview of Model Predictive Control*, IEEE Control System Magazine, Vol. 20, pp. 38–52, 2000.
5. Genceli, H., and Nikolaou, M., *Robust Stability Analysis of Constrained-Norm Model Predictive Control*, Journal of AIChE, Vol. 39, pp. 1954–1965, 1993.

6. Kothare, M. V., Balakrishnan, V., and Morari, M., *Robust Constrained Model Predictive Control Using Matrix Inequalities*, Automatica, Vol. 32, pp. 1361–1379, 1996.
7. Scokaert, P. O. M., and Mayne, D. Q., *Min-Max Feedback Model Predictive Control for Constrained Linear Systems*, IEEE Transactions on Automatic Control, Vol. 43, pp. 1136–1142, 1998.
8. Zhang, Z. Q., and Morari, M., *Robust Stability of Constrained Model Predictive Control*, Proceedings of the American Control Conference, pp. 379–383, 1993.
9. Bemporad, A., *Reducing Conservativeness in Predictive Control of Constrained Systems with Disturbances*, Proceedings of the 37th IEEE Conference on Decision and Control, pp. 1384–1389, 1998.
10. Jia, D., and Krogh, B. H., *Min-Max Feedback Model Predictive Control for Distributed Control with Communuication*, Proceedings of the American Control Conference, pp. 4507–4512, 2002.
11. Polak, E., *Optimization: Algorithms and Consistent Approximation*, Springer Verlag, New York, NY, 1997.
12. Vandenberghe, L., and Boyd, S., *Connections between Semi-Infinite and Semidefinite Programming*, Semi-Infinite Programming, Proceedings of the International Workshop on Semi-Infinite Programming, Edited by R. Reemtsen and J.J. Rueckmann, Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 277–294, 1998.
13. Todd, M. J., *Semidefinite Optimization*, Acta Numerica, Vol. 10, pp. 515–560, 2001.
14. Vandenberghe, L., and Boyd, S., *Semidefinite Programming*, SIAM Review, Vol. 38, pp. 49–95, 1996.
15. Boyd, S., El Ghaoul, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1994.
16. Blanchini, F., *Set Invariance in Control*, Automatica, Vol. 35, pp. 1747–1767, 1999.
17. Jia, D., *Distributed Coordination in Multi-Agent Control Systems through Model Predictive Control*, PhD Thesis, Department of Electrical and Computer Engineering, Carnegie Mellon University, 2003.
18. Kouvaritakis, B., Rossiter, J. A., and Schuurmans, J., *Efficient Robust Predictive Control*, IEEE Transactions on Automatic Control, Vol. 45, pp. 1545–1549, 2000.
19. Loefberg, J., *Minimax Approaches to Robust Model Predictive Control*, PhD Thesis, Department of Electrical Engineering, Linkoeping University, 2003.