Lecture #18

Introduction To Scheduling

18-348 Embedded System Engineering

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Electrical & Computer ENGINEERING

Carnegie Mellon

Sewer And Pipe Inspection Camera

http://www.wastewaterpr.com/releases/view/692/RIDGID-Introduces-SeeSnake-Laptop-Interface

Where Are We Now?

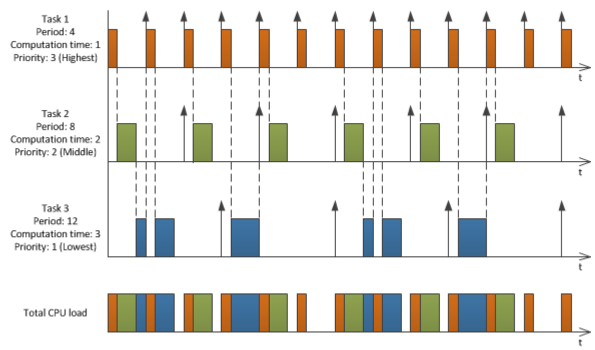
- **Where we've been:**
  - Interrupts
  - Context switching and response time analysis
  - Concurrency
- **Where we're going today:**
  - Scheduling
- **Where we're going next:**
  - Analog and other I/O
  - System booting, control, safety, …
  - In-class Test #2, Wed 20-April-2016
  - Final project due finals week. No final exam.

Preview

- **What's Real Time?**
- **Scheduling – will everything meet its deadline?**
  - Schedulability
  - 5 key Assumptions
- **Application of scheduling**
  - Static multi-rate systems
  - Dynamic priority scheduling: Earliest Deadline First (EDF) and Least Laxity
  - Static priority preemptive systems (Rate Monotonic Scheduling)
- **Related topics**
  - Blocking time
  - Sporadic tasks

Real Time Scheduling Overview

- Hard real time systems have a deadline for each periodic task
  - With an RTOS, the highest priority active task runs while others wait
  - System fault occurs every time a task misses a deadline
  - Mathematical analysis is accepted practice for ensuring deadlines are met
    - We'll build up to Rate Monotonic Analysis in this lecture



(Alexeev 2011, p. 5)

Schedulability

Meeting hard deadlines is one of the most fundamental requirements of a real-time operating system and is especially important in safety-critical systems. Depending on the system and the thread, missing a deadline can be a critical fault.

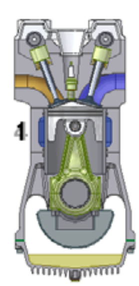
Rate monotonic analysis (RMA) is frequently used by system designers to analyze and predict the timing behavior of systems.

(Kleidermacher 2001 pg. 30)

Real Time Definitions

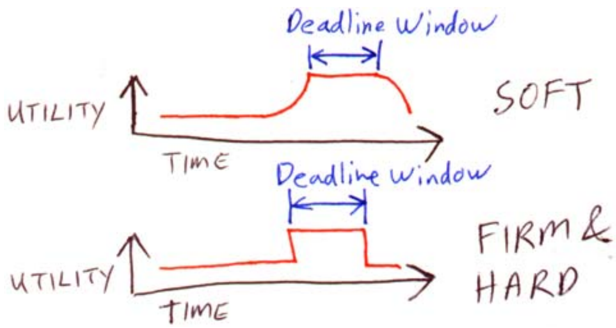
- **Reactive:**
  Computations occur in response to external events
  - Periodic events (*e.g.*, rotating machinery and control loops)
    - Most embedded computation is periodic
  - Aperiodic events (*e.g.*, button closures)
    - Often they can be "faked" as periodic (e.g., sample buttons at 10 Hz)

- **Real Time**
  - Real time means that correctness of result depends on both functional correctness and time that the result is delivered
  - Too *slow* is usually a problem
  - Too *fast* sometimes is a problem



Flavors Of Real Time

- **Soft real time**
  - Utility degrades with distance from deadline
- **Hard real time**
  - System fails if deadline window is missed
- **Firm real time**
  - Result has no utility outside deadline window, but system can withstand a few missed results



"Real Time" != "Really Fast"

- **"Real Time" != "Really Fast"**
  - It means not too fast and not too slow
  - Often the "not too slow" part is more difficult, but it's not the only issue
  - Also, a whole lot faster than you need to go can be wasteful overkill
  - Often, ability to be consistently on time is more important than "fast"
- **Consider what happens when a CPU goes obsolete**
  - Is it OK to write a software simulator on a really fast newer CPU?
    - Will timing be fast enough?
    - Will it be too fast?
    - Will it vary more than the old CPU?
  - What do designers actually do about this?

Types of Real-Time Scheduling

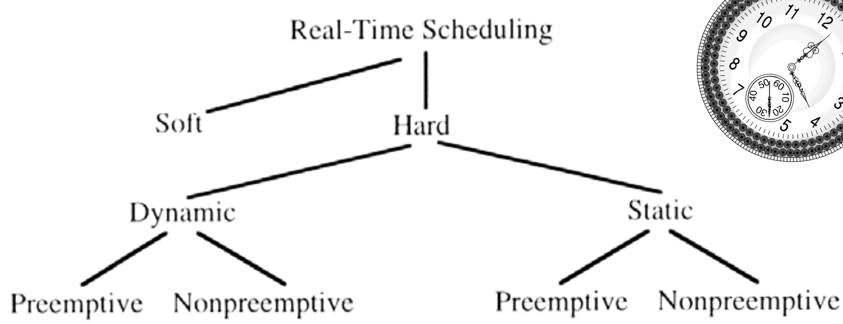


Figure 11.1: Taxonomy of real-time scheduling algorithms.

[Kopetz]

- **Dynamic vs. Static**
  - Dynamic schedule computed at run-time based on tasks really executing
  - Static schedule done at compile time for all *possible* tasks
- **Preemptive permits one task to preempt another one of lower priority**

Schedulability

- **NP-hard if there are any resource dependencies at all**
  - So, the trick is to put cheaply computed bounds/heuristics in place
    - Prove it definitely can't be scheduled
    - Find a schedule if it is easy to do so
    - Punt if you're in the middle somewhere

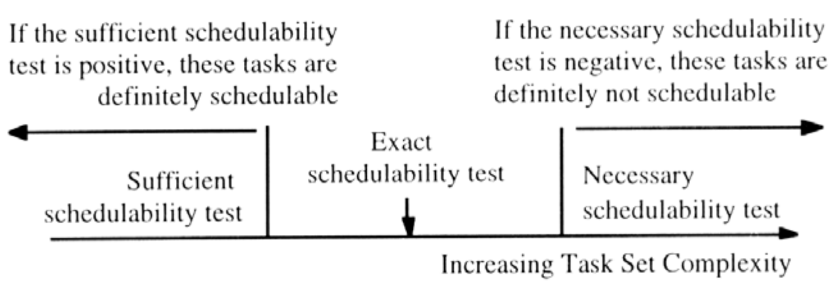
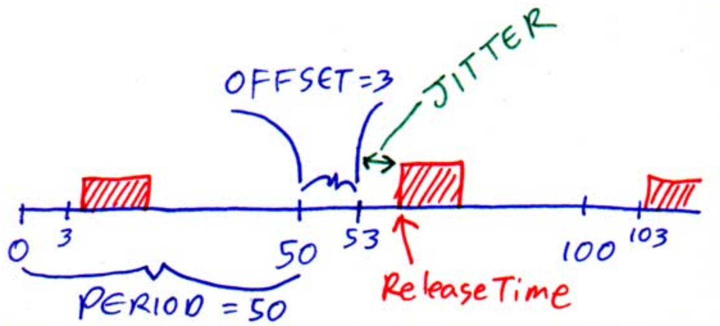


Figure 11.2: Necessary and sufficient schedulability test.

[Kopetz]

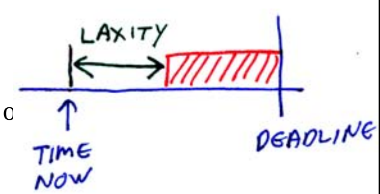
Periodic Tasks

- **"Time-triggered" (periodic) tasks are common in embedded systems**
  - Often via control loops or rotating machinery
- **Components to periodic tasks**
  - Period (e.g, 50 msec)
  - Offset past period (e.g., 3 msec offset/50 msec period -> 53, 103, 153, 203)
  - Jitter is random "noise" in task release time (*not* oscillator drift)
  - Release time is when task has its "ready to run" flag set
  - Release time_n= (n*period) + offset + jitter   ; assuming perfect time precision



Scheduling Parameters

- **Set of tasks $\{T_i\}$**
  - Periods p_i
  - Deadline d_i
    (completion deadline after task is queued)
  - Execution time c_i
    (amount of CPU time to complete)
  - Worst case latency to complete execution W_i
    - This is something we solve for, it's not a given
- **Handy values:**
  - Laxity $l_i = d_i - c_i$
    (amount of slack time before Ti *must* begin execution)
  - Utilization factor $\mu_i = c_i/p_i$ (portion c
    CPU used)



Major Assumptions

- **Five assumptions are the starting point for this area:**
  1. **Tasks $\{T_i\}$ are periodic, with hard deadlines and no jitter**
     - **Period is P_i**
  2. **Tasks are completely independent**
     - **B=0; Zero blocking time; no use of a mutex; interrupts never masked**
  3. **Deadline = period**
     - **$P_i = D_i$**
  4. **Computation time is known (use worst case)**
     - **C_i is always the same for each execution of the task**
  5. **Context switching is free (zero cost)**
     - **Executive takes zero overhead, and task switching has zero latency**

- **These assumptions are often not realistic**
  - But sometimes they are close enough in practice
  - Significantly relaxing these assumptions quickly becomes a grad school topic
    - We're going to show you the common special cases that are "easy" to use

Easy Schedulability Test

- **System is schedulable (i.e., it "works") if for all i, $W_i <= D_i$**
  - In other words, all tasks complete execution before their deadline

- **μ is processor utilization (fraction of time busy) must be less than 1**

$$\mu = \sum \frac{c_i}{p_i} \leq 1$$

  - "You can't use more that 100% of available CPU power!"

- **This is *necessary*, but not sufficient**
  - Sometimes even very low percent of CPU power used is still unschedulable
  - e.g., if blocking time exceeds shortest deadline, impossible to schedule system
  - e.g., several short-deadline tasks all want service at exactly the same time, but rest of time system is idle

Remember this? Multi-Rate Round Robin Approach

- **Simple brute force version**
  - Put some tasks multiple times in single round-robin list
  - But gets tedious with wide range in rates
- **More flexible version**
  - For each PCB keep:
    - Pointer to task to be executed
    - Period (number of times main loop is executed for each time task is executed) i.e., execute this task every k*th* time through main loop.
    - Current count – counts down from Period to zero, when zero execute task

```
typedef void (*pt2Function)(void);

struct PCB_struct
{ pt2Function Taskptr;   // pointer to task code
  uint8       Period;    // execute every kth time
  uint8       TimeLeft;  // starts at k, counts down
  uint8       ReadyToRun; // flag used later
};
PCB_struct PCB[NTASKS];  // array of PCBs
```


Remember this?

Time-Based Prioritized Cooperative Tasking

- **Assume `timer_ticks` is number of TCNT overflows recorded by ISR**

```
struct PCB_struct
{ pt2Function Taskptr;   // pointer to task code
  uint8       Period;    // Time between runs
  uint8       NextTime;  // next time this task should run
};
… init PCB structures etc. …

for(;;)
  { for (i = 0; i < NTASKS; i++)
    { if (PCB[i].NextTime < timer_ticks)
      {PCB[i].NextTime += PCB[i].Period; // set next run time
              // note – NOT  timer_ticks + Period  !!
       PCB[i].Taskptr();
       break;  // exit loop and start again at task 0
      }
    }
  }
```

- **This executes tasks in a particular order based on period and task #**
  - But, there is no guarantee that you will meet your deadlines in the general case!

Static Multi-Rate Periodic Schedule

- Assume <u>non-preemptive</u> system with <u>5 Restrictions</u>:
  1. Tasks $\{T_i\}$ are perfectly periodic
  2. B=0
  3. $P_i = D_i$
  4. Worst case C_i
  5. Context switching is free
- Consider least common multiple of periods p_i
  - This considers all possible cases of period phase differences
  - Worst case is time that is LCM of all periods
    - E.g., LCM(5,10,35) = 5 * 2 * 7 = 70
  - If you can figure out (somehow) how to schedule statically this, you win
    - Program in a static schedule that runs tasks in exactly that order at those times
    - Schedule repeats every LCM time period (e.g., every 70 msec for LCM=10)
    - This is a long-running computational problem for large task sets!
- Performance
  - Optimal if all tasks always run; can get up to **100% utilization** ($\mu = 1.00$)
  - If it runs once, it should always work

Example Static Schedule – Hand Positioned Tasks

Task #	Period (P_i)	Compute (C_i)
T1	5	1
T2	10	2
T3	15	2
T4	20	3
T5	25	4

Ensuring schedulability requires hand-selecting the start time of every task (not the same as the previous scheduler code)!

Start Time	Task #	C_i	Elapsed Time For T_i
0	T1	1	…
1	T5	4	…
5	T1	1	5-0=5
6	T2	2	…
8	T3	2	…
10	T1	1	10-5=5
11	T4	3	…
14	Idle	1	n/a
15	T1	1	15-10=5
16	T2	2	16-6=10
18	Idle	2	n/a
20	T1	1	20-15=5
21	Idle	2	n/a
23	T3	2	23-8=15
25	T1	1	25-20=5
26	T2	2	26-16=10

Preemptive, Prioritized Schedulability

- **To avoid missing deadlines, *necessary* for all the tasks to fit**
  - Time to complete task T_j is **W_j**
  - (i.e., we need to find out if this task set is "schedulable?")

$$\forall_j : W_j \overset{?}{\leq} P_j$$

  - If true, we are schedulable; if false we aren't
  - Note that this is W = time to complete task
    - It's ***not*** R = time to start execution of task (response time)
    - For cooperative scheduling, $W_i = R_i + C_i$
    - BUT, for preemptive scheduling W can be longer because of additional preemptions

- **In other words, schedulable if task completes before its period**
  - Always true if time to complete task T_j doesn't exceed period
  - True because we assumed that $\mathbf{P_i = D_i}$

What's Latency For Preemptive Tasks?

- **For the same 5 assumptions**
  - And prioritized tasks (static priority – priority never changes)
    - Note that equation includes execution time of task, not just response time

$$W_{m,0} = B + C_0$$

$$W_{m,i+1} = B + \sum_{j=0}^{j=m} \left(\left\lfloor \frac{W_{m,i}}{P_j} + 1 \right\rfloor C_j \right)$$

  - Note that in this math we are including the C term for task m in the summation
  - Highest priority task has only blocking time B as latency
  - Start the recursion with task 0, which could always execute first
  - Schedulable if: $\forall_j : W_j \leq P_j$

- **This math is complex, and easy to get wrong**
  - Is there an easier way to make sure we can't mess this up?

Remember the Major Assumptions

- **Five assumptions throughout this lecture**
  1. **Tasks $\{T_i\}$ are perfectly periodic**
  2. **B=0**
  3. **$P_i = D_i$**
  4. **Worst case C_i**
  5. **Context switching is free**

EDF: Earliest Deadline First

- **Assume a *preemptive* system with *dynamic* priorities, and { same 5 restrictions }**

- **Scheduling policy:**
  - Always execute the task with the nearest deadline
    - Priority changes on the fly!
    - Results in more complex run-time scheduler logic

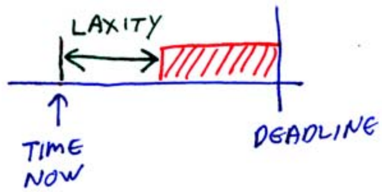
- **Performance**
  - Optimal for uniprocessor (supports up to 100% of CPU usage in all situations)
    - If it can be scheduled – but no guarantee that can happen!
    - Special case where it works is very similar to case where Rate Monotonic can be used:
      - » Each task period must equal task deadline
      - » But, still pay run-time overhead for dynamic priorities

  - If you're overloaded, ensures that a lot of tasks don't complete
    - Gives everyone a chance to fail at the expense of the later tasks

Least Laxity

- **Assume a *preemptive* system with *dynamic* priorities, and { same 5 restrictions }**

- **Scheduling policy:**
  - Always execute the task with the smallest laxity $l_i = d_i - c_i$



- **Performance:**
  - Optimal for uniprocessor (supports up to 100% of CPU usage in all situations)
    - Similar in properties to EDF
    - If it can be scheduled – but no guarantee that can happen!
  - A little more general than EDF for multiprocessors
    - Takes into account that slack time is more meaningful than deadline for tasks of mixed computing sizes
  - Probably more graceful degradations
    - Laxity measure permits dumping tasks that are hopeless causes

EDF/Least Laxity Tradeoffs

- **Pro:**
  - If it works, it can get 100% efficiency (on a uniprocessor)
  - Does not restrict task periods
  - Special case works if, for each task, Period = Deadline

- **Con:**
  - It is not always feasible to prove that it will work in all cases
    - And having it work for a while doesn't mean it will always work
  - Requires dynamic prioritization
  - EDF has bad behavior for overload situations (LL is better)
  - The laxity time hack for global priority has limits
    - May take too many bits to achieve fine-grain temporal ordering
    - May take too many bits to achieve a long enough time horizon

- **Recommendation:**
  - Avoid EDF/LL if possible
    - Because you don't know if it will really work in the general case!
    - And the special case doesn't buy you much, but comes at expense of dynamic priorities

Remember the Major Assumptions

- **Five assumptions throughout this lecture**
  1. **Tasks $\{T_i\}$ are perfectly periodic**
  2. **B=0**
  3. **$P_i = D_i$**
  4. **Worst case C_i**
  5. **Context switching is free**

- **Problems with previous approaches**
  - Static scheduling – can be difficult to find a schedule that works
  - EDF & LL – run-time overhead of dynamic priorities

  - Wanted: an easy rule for scheduling with:
    - Static priorities
    - Guaranteed schedulability

Rate Monotonic Scheduling

1. **Sort tasks by period (i.e., by "rate")**
2. **Highest priority goes to task with shortest period (fastest rate)**
   - Tie breaking can be done by shortest execution time at same period
3. **Use prioritized preemptive scheduler**
   - Of all ready to run tasks, task with fastest rate gets to run

- **Static priority**
  - Priorities are assigned to tasks at design time; priorities don't change at run time

- **Preemptive**
  - When a high priority task becomes ready to run, it preempts lower priority tasks
  - This means that ISRs have to be so short and infrequent that they don't matter

- **Variation: Deadline Monotonic**
  - Use min(period, deadline) to assign priority rather than just period
  - Works the same way, but handles tasks with deadlines shorter than their period

Rate Monotonic Scheduling (RMS)

- **Assume a *preemptive* system with *static* priorities, N tasks, and { same 5 restrictions } +**

$$\mu = \sum \frac{c_i}{p_i} \le N(\sqrt[N]{2} - 1) \qquad ; \mu \le \ln(2) \approx 0.693 \text{ for large N}$$

("CPU load less than about 70%")

- **Why not 100%?**
  - Two tasks with slightly different periods can drift in and out of phase
  - At just the wrong phase difference, there may not be time to meet deadlines
- **Performance:**
  - Provides a *guarantee* for schedulability with CPU load of ~70%
    - Even with *arbitrarily* selected task periods
    - Can do better if you know about periods & offsets
  - BUT – if you load CPU more than 69.3%, you might miss deadlines!

Example of a Missed Deadline at 79% CPU Load

TOTAL CPU LOAD:	79%	for all tasks		
	Task 1	Task 2	Task 3	Task 4
Period:	19	24	29	34
Compute:	5	5	5	5
Utilization:	26.3%	20.8%	17.2%	14.7%

Time	Task 1	Task 2	Task 3	Task 4	Running Task
0	RUN 1				1
1	RUN 2				1
2	RUN 3				1
3	RUN 4				1
4	RUN 5				1
5	..sleep..	RUN 1			2
6	..sleep..	RUN 2			2
7	..sleep..	RUN 3			2
8	..sleep..	RUN 4			2
9	..sleep..	RUN 5			2
10	..sleep..		RUN 1		3
11	..sleep..		RUN 2		3
12	..sleep..		RUN 3		3
13	..sleep..		RUN 4		3
14	..sleep..		RUN 5		3
15	..sleep..			RUN 1	4
16	..sleep..			RUN 2	4
17	..sleep..			RUN 3	4
18	..sleep..			RUN 4	4
19	RUN 1				1
20	RUN 2				1
21	RUN 3				1
22	RUN 4				1
23	RUN 5				1
24		RUN 1			2
25		RUN 2			2
26		RUN 3			2
27		RUN 4			2
28		RUN 5			2
29			RUN 1		3
30			RUN 2		3
31			RUN 3		3
32			RUN 4		3
33			RUN 5		3

RUN 5???

No Place To Schedule RUN 5
Task 5 Misses Its Deadline of 34

- **Task 4 misses deadline**
  - This is the worst case launch time scenario
- **Missed deadlines can be difficult to find in system testing**
  - 5 time units per task is worst case
    - Average case is often a bit lighter load
  - Tasks only launch all at same time once every 224,808 time units

LCM(19,24,29,34) = 224,808
(LCM = Least Common Multiple)

Harmonic RMS

- **In most real systems, people don't want to sacrifice 30% of CPU**
  - Instead, use harmonic RMS
- **Make all periods harmonic multiples**
  - P_i is evenly divisible by all shorter P_j
  - This period set is harmonic: {5, 10, 50, 100}
    - <u>10</u> = <u>5</u>*2; <u>50</u> = <u>10</u>*<u>5</u>; <u>100</u> = <u>50</u>*2; <u>100</u> = <u>10</u>*<u>5</u>*2
  - This period set is ***not*** harmonic: {3, 5, 7, 11, 13}
    - 5 = 3 * 1.67 *(non-integer)*, etc.
- **If all periods are harmonic, works for <u>CPU load of 100%</u>**
  - Harmonic periods can't drift in and out of phase – avoids worst case situation

$$\mu = \sum \frac{c_i}{p_i} \le 1 \quad ; \quad \forall_{p_j < p_i} \{ p_j \text{ evenly divides } p_i \}$$

Practical Harmonic Deadline Monotonic Scheduling

- **This is what you should do in most smaller embedded control systems**
  - Assumes you need a preemptive scheduler
- **Use Min(period,deadline) as the scheduling logical "period"**
  - Ensures that deadline will be met even if shorter than period
  - But, set aside resources just as if tasks really were repeating at that period
  - This is the part that makes it "deadline" monotonic
- **Use harmonic multiples of logical period**
  - Every shorter period is a factor of every longer period (e.g., 1, 10, 100, 1000)
  - Avoids worst case of slightly out-of-phase periods that all clump together at just the wrong time
  - Speed up some tasks if needed to get harmonic multiples
    - E.g., {1, 5, 11, 20} => {1, 5, 10, 20}
    - Results in lower CPU requirement *even though* some tasks run faster!
- **Watch out for blocking!**

Example Deadline Monotonic Schedule

Task #	Period (P_i)	Deadline (D_i)	Compute (C_i)
T1	5	15	1
T2	16	23	2
T3	30	6	2
T4	60	60	3
T5	60	30	4

Task #	Priority	μ
T1	1	1/5 = 0.200
T3	2	2/6 = 0.333
T2	3	2/16 = 0.125
T5	4	4/30 = 0.133
T4	5	3/60 = .05
	TOTAL:	0.841

$$\mu = \sum \frac{c_i}{p_i} \le N(\sqrt[N]{2} - 1) \qquad ; \mathrm{N} = 5$$

$$\mu = 0.841 \quad (not \le) \quad 0.743$$

Not Schedulable!

(might be OK with fancy math)

Example Harmonic Deadline Monotonic Schedule

Task #	Period (P_i)	Deadline (D_i)	Compute (C_i)
T1	5	15	1
T2	*15*	23	2
T3	30	*5*	2
T4	60	60	3
T5	60	30	4

Task #	Priority	μ
T1	1	1/5 = 0.200
T3	2	2/5 = 0.400
T2	3	2/15 = 0.133
T5	4	4/30 = 0.133
T4	5	3/60 = .05
	TOTAL:	0.916

$$\mu = \sum \frac{c_i}{p_i} \le 1 \qquad ; H\text{armonic periods } \{5, 15, 30, 60\}$$

$$\mu = 0.916 \quad \le \quad 1$$

Schedulable, even though usage is higher!

Handling Non-Zero Blocking

- **Rate monotonic, but task blocking can occur**
  - B_k is time task k can be blocked (e.g., interrupts masked by lower prio task)
  - For highest priority task
    - Can ignore lower priority tasks, because we are preemptive
    - But, need to handle blocking time (possibly caused by lower priority task)

$$\mu_1 = \left(\frac{c_1}{p_1}\right) + \frac{B_1}{p_1} \leq 1(\sqrt[1]{2} - 1)$$

  - For 2nd highest priority task
    - Can ignore lower priority tasks, because we are preemptive
    - Have to account for highest priority task preempting us
    - Need to handle blocking time
      - » Possibly caused by lower priority task
      - » But, can't be caused by higher priority task (since that preempts us anyway)
      - » Does this sound a lot like the reasoning behind ISR scheduling???

$$\mu_2 = \left(\frac{c_1}{p_1}\right) + \left(\frac{c_2}{p_2}\right) + \frac{B_2}{p_2} \leq 2(\sqrt[2]{2} - 1)$$

Rate Monotonic With Blocking

- **Rate monotonic, but task blocking can occur**
  - B_k is blocking time of task k (time spent stalled waiting for resources)

$$\forall k;\ \mu_k = \sum_{i \leq k} \mu_i = \sum_{i \leq k} \left(\frac{c_i}{p_i}\right) + \frac{B_k}{p_k} \leq k(\sqrt[k]{2} - 1) \quad \approx 0.7 \text{ for large k}$$

[Sha et al. 1991]

  - Worst case blocking time for each task counts as CPU time for scheduling
  - Note that B includes all interrupt masking (ISRs *__and__* tasks waiting for CLI)
  - Harmonic periods make right hand side 100%, as before
  - Need on a per-task basis because blocking time can be different for each task

- **Performance:**
  - In worst case, time waiting while blocked is counted as burning additional CPU or network time
  - This is yet another reason to use skinny ISRs!
  - If low priority task gets a mutex needed by a hi prio task, it extends B!
  - If RTOS takes a while to change tasks, that counts as blocking time too

Applied Deadline Monotonic With Blocking

- **Use min(period, deadline) for each task as logical period**
  - Use harmonic logical periods
  - Assign tasks by priority
  - Otherwise, same as for deadline monotonic

- **For each task,**

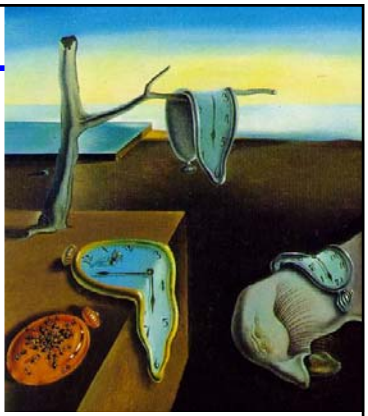
$$\mu_1 = \left(\frac{c_1}{p_1}\right) + \frac{B_1}{p_1} \le 1$$

$$\mu_2 = \left(\frac{c_1}{p_1}\right) + \left(\frac{c_2}{p_2}\right) + \frac{B_2}{p_2} \le 1$$

$$\mu_3 = \left(\frac{c_1}{p_1}\right) + \left(\frac{c_2}{p_2}\right) + \left(\frac{c_3}{p_3}\right) + \frac{B_3}{p_3} \le 1$$

$$\forall k; \mu_k = \sum_{i \le k} \mu_i = \sum_{i \le k} \left(\frac{c_i}{p_i}\right) + \frac{B_k}{p_k} \le 1 \text{ ; for harmonic periods}$$

But Wait, There's More



- **WHAT IF:**
  1. Tasks $\{T_i\}$ are NOT periodic
     – Use maximum fastest inter-arrival time
  2. Tasks are NOT completely independent
     – Worry about dependencies **(another lecture)**
  3. Deadline NOT = period
     – Use Deadline monotonic
  4. Worst case computation time c_i isn't known
     – Use worst case computation time, if known
     – Build or buy a tool to help determine Worst Case Execution Time (WCET)
     – Turn off caches and otherwise reduce variability in execution time
  5. Context switching is free (zero cost)
     – Gets messy depending on assumptions
     – Might have to include scheduler as task
     – Almost always need to account for blocking time B

Review

- **Real time definitions**
  - Hard, firm, soft
- **Scheduling – will everything meet its deadline?**
  - $\mu \le 1$
  - All $W_i \le P_i$
- **Application of scheduling**
  - Static multi-rate systems
  - Rate Monotonic Scheduling
    - $\mu \le 1$ *if harmonic periods;* else more like 70%
    - Works by assigning priorities based on periods (fastest tasks get highest prio
- **Related topics**
  - Earliest Deadline First (EDF) and Least Laxity
  - Blocking
  - Sporadic server

Review

- Five Standard Assumptions (***memorize them in exactly these words – notes sheet too***):
  1. **Tasks $\{T_i\}$ are perfectly periodic**
  2. **B=0**
  3. **$P_i = D_i$**
  4. **Worst case C_i**
  5. **Context switching is free**
- **Statically prioritized task completion times:**

$$W_{m,0} = C_0$$

$$W_{m,i+1} = B + \sum_{j=0}^{j=m} \left(\left\lfloor \frac{W_{m,i}}{P_j} + 1 \right\rfloor C_j \right)$$

Review

- **Schedulability bound for Rate Monotonic with Blocking**

$$\mu_1 = \left(\frac{c_1}{p_1}\right) + \frac{B_1}{p_1} \le 1$$

$$\mu_2 = \left(\frac{c_1}{p_1}\right) + \left(\frac{c_2}{p_2}\right) + \frac{B_2}{p_2} \le 1$$

$$\mu_3 = \left(\frac{c_1}{p_1}\right) + \left(\frac{c_2}{p_2}\right) + \left(\frac{c_3}{p_3}\right) + \frac{B_3}{p_3} \le 1$$

$$\forall k; \mu_k = \sum_{i \le k} \mu_i = \sum_{i \le k} \left(\frac{c_i}{p_i}\right) + \frac{B_k}{p_k} \le 1 \text{ ; for harmonic periods}$$