Spectral Analysis

Eigenvalues / Eigenvectors are a valuable tool:

- resonance of a system
- robust classifiers
- long-term stability of systems

They capture some "essence" of a matrix.

Example

$$A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix} \quad \ldots \text{ok}$$

$$A \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \end{bmatrix} = 6 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \ldots \text{hmm?}$$

$$A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \ldots \text{hmm?}$$

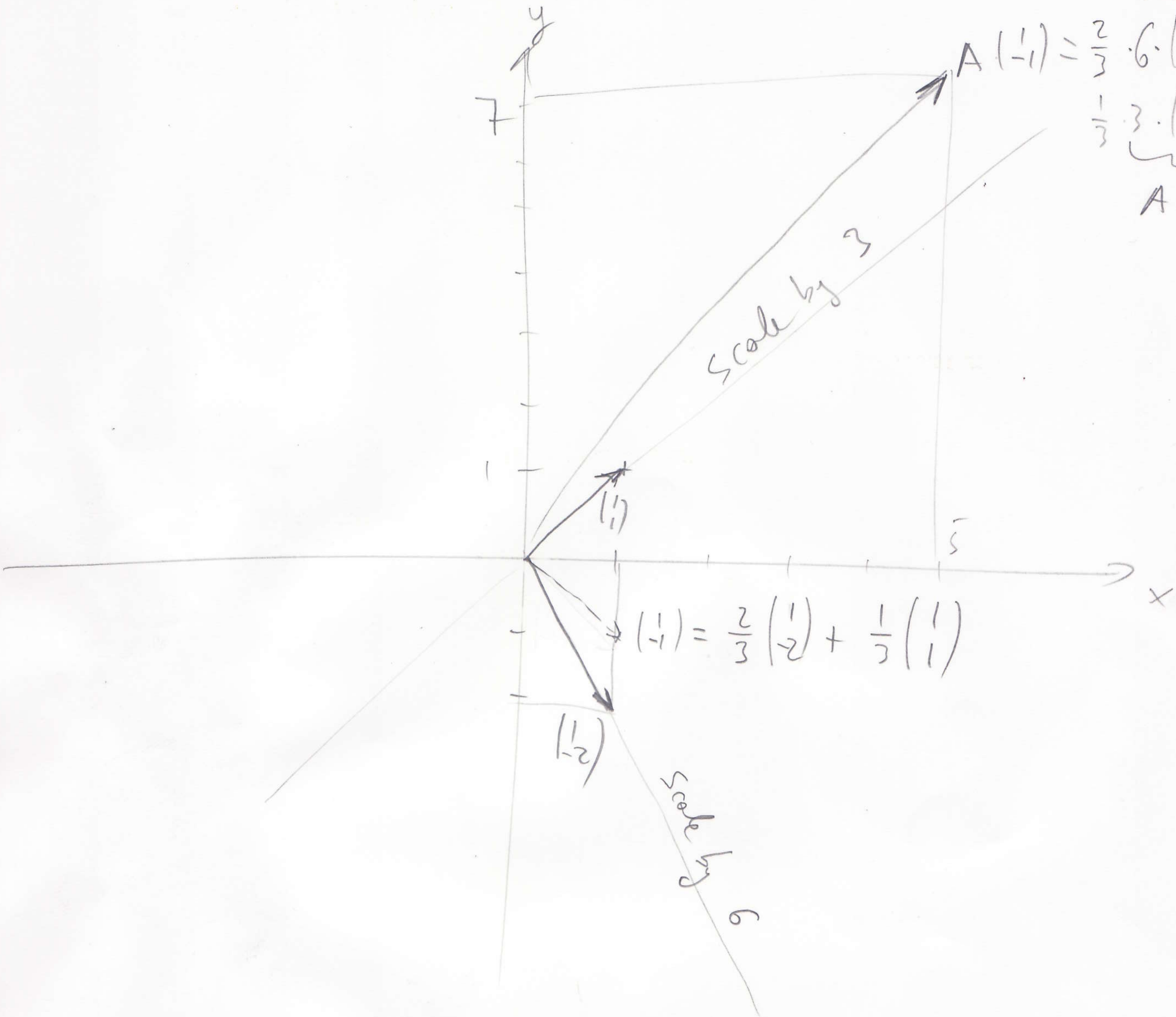
$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}$$

$$(\cdot)^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}}_{\text{basis transformation}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

②



$$A\begin{pmatrix}1\\-1\end{pmatrix} = \frac{2}{3}\cdot\overbrace{6\cdot\begin{pmatrix}1\\-2\end{pmatrix}}^{A\cdot\begin{pmatrix}1\\-2\end{pmatrix}} + \frac{1}{3}\cdot\underbrace{3\cdot\begin{pmatrix}1\\1\end{pmatrix}}_{A\cdot\begin{pmatrix}1\\1\end{pmatrix}}$$

$$\begin{pmatrix}1\\-1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}1\\-2\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\end{pmatrix}$$

Definition

A is $n \times n$, λ is a scalar, x is a non-zero vector

The λ for which $Ax = \lambda x$ are the eigenvalues of A. x is the eigenvector to λ

Characteristic Polynomial

$(\lambda I - A) \cdot x = 0$ characterizes λ

for that equation to have non-trivial solutions:

$\text{rank}(\lambda I - A) < n \iff |\lambda I - A| = 0$ ← determinant

Characteristic Polynomial

$$\Delta(\lambda) = |\lambda I - A|$$

Characteristic equation

$$\Delta(\lambda) = |\lambda I - A| = 0$$

roots of $\Delta(\lambda)$ are the eigenvalues of A

Example:

$$\left| \begin{pmatrix} \lambda-4 & 1 \\ 2 & \lambda-5 \end{pmatrix} \right| = (\lambda-4)(\lambda-5) - 2 = \lambda^2 - 9\lambda + 18 = (\lambda-3)(\lambda-6)$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = 6$$

Details: characteristic polynomial

- A is $n \times n$, $|\lambda I - A| = \Delta(\lambda)$
- $\Delta(\lambda)$ has degree n
- for $A \in \mathbb{R}^{n\times n}$, $\Delta(\lambda)$ has real coefficients
- leading coefficient is 1

Number of eigenvalues:

$n \times n$ matrix A has n eigenvalues
- not necessarily distinct
- if matrix is real, then eigenvalues are real or complex conjugates

proof sketch: see polynomials / complex numbers
fundamental theorem of algebra

Eigenvectors

The eigenspace for eigenvector λ_i is given by the null space

$$\mathcal{N}(\lambda_i I - A)$$

Thus, eigenvector v_i for eigenvalue λ_i is found by solving the linear system

$$A v_i = \lambda_i v_i$$

- Eigenvectors are not unique, as for v_i eigenvector, αv_i is also an eigenvector
- Often normalized to $\begin{pmatrix} 1 \\ x \\ x \end{pmatrix}$ or $\|v_i\| = 1$

The eigenspace for each eigenvalue can be 1 or higher dimensional: depends on $\operatorname{rank}(\lambda_i I - A)$

- if $\operatorname{rank}(\lambda_i I - A) = n-1 \longrightarrow$ 1 dim eigenspace
- in general: if the multiplicity of root λ_i is m_i, then
  - there is at least 1 l.i. eigenvector for λ_i
  - there are no more than m_i l.i. eigenvectors

Details are involved

Example

$$A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}, \quad \lambda_1 = 6$$

$$6 I_2 - A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \quad \text{rank} = 1$$

$$\begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = 6 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$4u - v = 6u \quad | -4u$$

$$v = -2u$$

pick $u = 1 \Rightarrow v = -2 \Rightarrow V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

n Distinct Eigenvalues

- Special case.
- multiplicity of all eigenvalues is 1
- eigenvectors of distinct eigenvalues are linear independent

Diagonalization

A is a $n \times n$ matrix
$\lambda_1, \dots, \lambda_n$ are n distinct eigenvalues
$v_1, \dots, v_n$ are the respective eigenvectors

$V := [v_1 \ \dots \ v_n]$

$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Then:

$$A = V \Lambda V^{-1}$$

and

$$\Lambda = V^{-1} A V$$

V, V^{-1} are basis transformations to express the linear transfor given by A in terms of the eigenvectors

Power of a Matrix

$$A = V\underline{\Lambda}V^{-1}$$

$$A^2 = V\underline{\Lambda}V^{-1} \cdot V\underline{\Lambda}V^{-1} = V\underline{\Lambda}^2V^{-1}$$

$$A^p = V\underline{\Lambda}^pV^{-1}$$

Inverse of a matrix

$$A^{-1} = (V\underline{\Lambda}V^{-1})^{-1} = V\underline{\Lambda}^{-1}V^{-1}$$

Determinant of a matrix

$$|A| = |V||\underline{\Lambda}||V^{-1}| = |\underline{\Lambda}| = \prod_{i=1}^{n} \lambda_i$$

Practical issue:

- To compute $\Delta(\lambda)$ the determinant is needed
- need to root a n'th order polynomial

$\rightarrow$ for $n \geq 5$ no closed form, expensive in practice, numerical problems

Special matrices

Orthogonal matrices

A is $n \times n$, real, orthogonal if $A^{-1} = A^T$

$|A| = \pm 1$, but from $|A| = \pm 1$ orthogonality does not follow

Rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$|R(\theta)| = 1$$

$$R(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(\theta)^T$$

spectrum is on unit circle, i.e. $|\lambda|^2 = 1$

Symmetric matrices

for A $n \times n$, symmetric, i.e. $A^T = A$:

- all eigenvalues and eigenvectors are real-valued
- n linearly independent eigenvectors $v_1, \dots, v_n$
- $v_1, \dots, v_n$ can be chosen to be orthonormal
- A is diagonalized by an orthonormal transformation

$$\Lambda = V^T A V$$

Positive Definite Matrices

A $n \times n$ is positive definite if

$$\forall x \in \mathbb{R}^n, x \neq 0: x^T A x > 0$$

similar: semi-definite, negative (semi) definite

$$A \succ 0$$

Unitary Matrices

square matrix A is unitary iff $AA^H = A^H A = I$

all eigenvalues of unitary matrices are on the unit circle: $|\lambda|^2 = 1$

Hermitian Matrices

Hermitian matrix: $A = A^H$

All eigenvalues are real-valued

Similarity

A and B are similar iff $\exists$ S such that

$$A = S^{-1} B S$$

Then:

- $|A| = |B|$
- A and B have the same eigenvalues
- $tr(A) = tr(B)$

However, eigenvectors, the 4 subspaces change with similarity transform