Second Order Ordinary Difference Equations

General form (IVP)

$$a_2 x[k+2] + a_1 x[k+1] + a_0 x[k] = f[k], \quad k = 0, 1, 2, \ldots$$

$$x[0] = x_0$$
$$x[1] = x_1$$

Canonical form

$$x[k+2] + a_1 x[k+1] + a_0 x[k] = f[k], \quad k = 0, 1, 2 \ldots$$

$$x[0] = x_0$$
$$x[1] = x_1$$

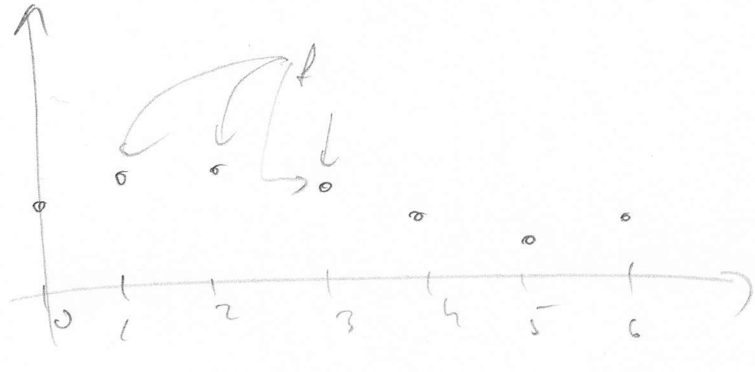
Solution

function $h[k]$ that leads to identity when plugged into ODE and obeys i.c.

Existence: family of functions w. 2 parameters for ODE $x_g[k]$

Uniqueness: $x[k]$ solution to IVP is unique

Plot of DT ODE



Structure of Solution

$x_g[k]$ general solution has form

$$x_g[k] = x_h[k] + x_p[k]$$

- $x_h[k]$ is homogeneous solution
  solution to $x[k+2] + a_1 x[k+1] + a_0 x[k] = 0$
  2 free parameters

- $x_p[k]$ is particular solution
  one solution to $x[k+2] + a_1 x[k+1] + a_0 x[k] = f[k]$
  no parameter

Linearity w.r.t. initial condition

Homogeneous IVP : with solution $x_h[k]$:

$$x[k+2] + a_1 x[k+1] + a_0 x[k] = 0 \qquad k \geq 0,$$

$x[0] = x_0$

$x[1] = x_1$

$x_{h_1}[k]$ solution to IVP with $x_{h_1}[0] = x_{0_1}$, $x_{h_1}[1] = x_{1_1}$

$x_{h_2}[k]$ solution to IVP with $x_{h_2}[0] = x_{0_2}$, $x_{h_2}[1] = x_{1_2}$

if $\quad x_0 = \alpha_1 x_{0_1} + \alpha_2 x_{0_2}$

$\quad\quad x_1 = \alpha_1 x_{1_1} + \alpha_2 x_{1_2}$

then

$$x_h[k] = \alpha_1 x_{h_1}[k] + \alpha_2 x_{h_2}[k]$$

Linearity w.r.t. input

For ODE: particular solution $x_p[k]$

$$x[k+2] + a_1 x[k+1] + a_0 x[k] = f[k]$$

Setup:

ODE 1:

$$x[k+2] + a_1 x[k+1] + a_0 x[k] = f_1[k]$$

ODE 2:

$$x[k+2) + a_1 x[k+1] + a_0 x[k] = f_2[k]$$

If

$$f[k] = \alpha_1 f_1[k] + \alpha_2 f_2[k]$$

Then

$$x_p[k] = \alpha_1 x_{p_1}[k] + \alpha_2 x_{p_2}[k]$$

Four Step solution

1) Homogeneous solution

$x_h[k]$: - characteristic polynomial

- root polynomial $\rightarrow$ solve $x_h[k]$

- roots = eigenvalues, characteristic values modes, natural frequencies

2) Particular solution

find a $x_p[k]$

- guessing method
- variation of constants
- separation of variables

3) General solution

$$x_g[k] = x_h[k] + x_p[k]$$

4) Impose i.c.

$x_g[0] = x_0$

$x_g[1] = x_1$ $\longrightarrow$ Solve for the 2 parameters

DT Homogeneous ODEs and IVP

Setup:

$$x[k+2] + a_1 x[k+1] + a_0 x[k] = 0, \quad x[0] = x_0, \; x[1] = x_1, \; k = 0, 1, \ldots$$

Example 1

$$x[k+2] - 2x[k+1] + 2x[k] = 0, \quad k = 0, 1, \ldots$$

$$x[0] = 1$$

$$x[1] = -1$$

Step 1 Homogeneous Solution

Ansatz: $x_h[k] = \alpha \rho^k$

Substitute into ODE:

$$\alpha \rho^{k+2} - 2\alpha\rho^{k+1} + 2\alpha\rho^k = 0$$

$$\alpha\rho^k(\rho^2 - 2\rho + 2) = 0$$

$$\Delta(\rho) = \rho^2 - 2\rho + 2$$

$$\Rightarrow \rho_{1,2} = 1 \pm j$$

The two modes of the ODE:

$$(1+j)^k \quad , \quad (1-j)^k \quad , \quad k=0,1,\dots$$

$$\Rightarrow \underline{x_h[k] = \left(c_1(1+j)^k + c_2(1-j)^k\right)}$$

Remark: get $\Delta(\rho)$ by inspection:

$$x[k+2] \to \rho^2$$
$$x[k+1] \to \rho$$
$$x[k] \to \rho^0 = 1$$

Step 2 Particular solution

$$x_p[k] \equiv 0$$

Step 3 General solution

$$x_g[k] = x_p[k] + x_h[k] = x_h[k]$$

Step 4 : IVP

Evaluate general solution at 0, 1,

$$x_g[0] = \boxed{c_1 + c_2 = 1}$$
$$x_g[1] = \boxed{c_1(1+j) + c_2(1-j) = -1}$$

$\Rightarrow$ order 2 linear system

Solve linear system:

$$c_1 = \frac{1}{2} + j$$

$$c_2 = \frac{1}{2} - j$$

$$x[k] = \left(\frac{1}{2} + j\right)(1+j)^k + \left(\frac{1}{2} - j\right)(1-j)^k$$

However, $x[k]$ is a real function.

$$\left(\left(\frac{1}{2}+j\right)(1+j)^k\right) = \left(\left(\frac{1}{2}-j\right)(1-j)^k\right)^*$$

by rules for $(\cdot)^*$

$$\Longrightarrow \quad x[k] = 2\,\mathrm{Re}\left(\left(\frac{1}{2}+j\right)(1+j)^k\right)$$

- convert to polar for multiplication/exponentiation

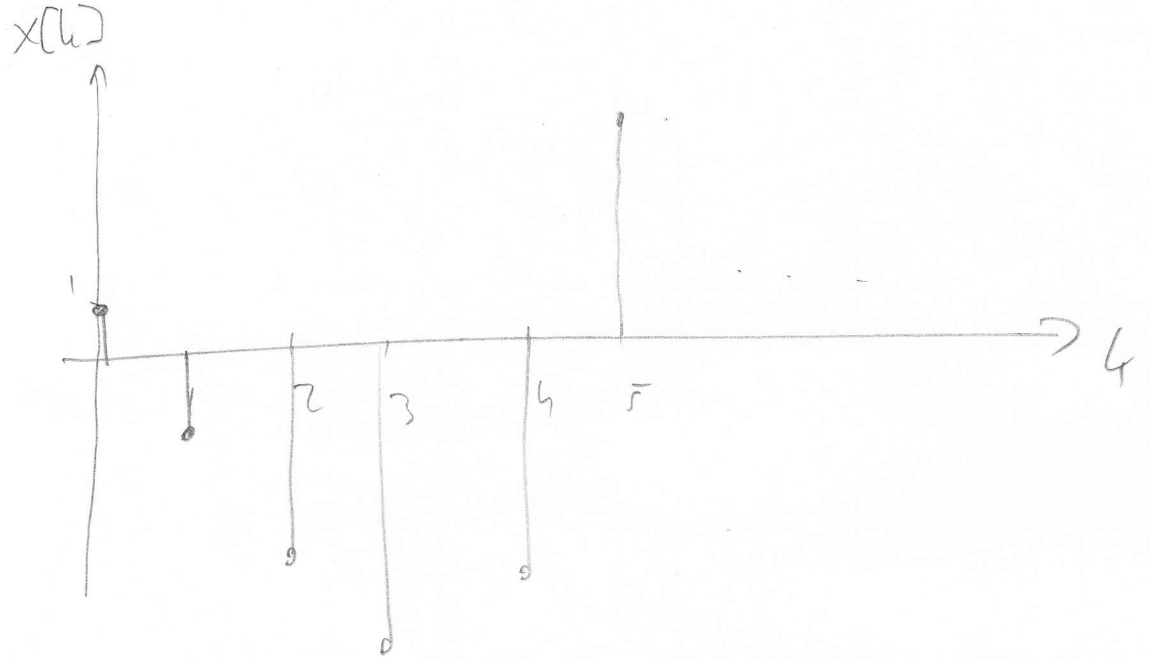
$$\frac{1}{2} + j = \frac{\sqrt{5}}{2}\, e^{j\,63.43^\circ}$$

$$1 + j = \sqrt{2}\, e^{j\frac{\pi}{4}}$$

$$\Rightarrow \quad x[k] = 2\,\mathrm{Re}\left(\frac{\sqrt{5}}{\cancel{2}}\, e^{j\,63.43^\circ}\, \sqrt{2}\, e^{j\frac{k\pi}{4}}\right)$$

$$x[k] = \sqrt{10}\,\cos\left(\frac{k\pi}{4} + 63.43^\circ\right)$$

Remark: $|\rho_1| = |\rho_2| > 1 \Rightarrow$ unstable



General Case / Summary

$$x[k+2] + a_1 x[k+1] + a_0 x[k] = 0, \quad k = 0, 1, 2, \ldots$$

$x[0] = x_0$

$x[1] = x_1$

Step 1 Homogeneous solution

Characteristic polynomial

$$\Delta(\rho) = \rho^2 + a_1 \rho + a_0$$

$$\Rightarrow \rho_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}, \quad \text{here, } \rho_1 \neq \rho_2$$

modes: $x_{h_1}[k] = \alpha_1 \rho_1^k$, $x_{h_2}[k] = \alpha_2 \rho_2^k$

$$\Rightarrow \underline{x_h[k] = c_1 \rho_1^k + c_2 \rho_2^k}, \quad k = 0, 1, 2,$$

two free parameters

Step 2: particular solution $x_p[k] \equiv 0$

Step 3: general solution $x_g[k] = x_h[k]$

Step 4: IVP

evaluate $x_h[k]$ at 0 and 1 $\Rightarrow$ lin system

$$\begin{aligned} c_1 + c_2 &= x_0 \\ \rho_1 c_1 + \rho_2 c_2 &= x_1 \end{aligned} \Rightarrow c_1, c_2$$

$\Longrightarrow$ find real representation (lots of algebra)

Closed Form Solution

$$x[k] = c_1 |\rho_1|^k \cos(\Omega_1 k - c_2)$$

for $\rho_1 = |\rho_1| e^{j\Omega_1}$ we get

$$c_1 = \frac{\sqrt{(x_1 - |\rho_1| \cos(\Omega_1) \cdot x_0)^2 + (|\rho_1| \sin(\Omega_1) \cdot x_0)^2}}{|\rho_1| \cdot |\sin \Omega_1|}$$

$$c_2 = \tan^{-1}\left(\frac{x_1 - |\rho_1| \cos(\Omega_1) \cdot x_0}{|\rho_1| \sin(\Omega_1) x_0}\right)$$

Note: Use of Heaviside function

we often use $u[k] = \begin{cases} 0, & k = -1, -2, \dots \\ 1, & k = 0, 1, \dots \end{cases}$

to capture the $k = 0, 1, 2, \dots$ constraint

e.g. $x[k] = (c_1 \rho_1^k + c_2 \rho_2^k)\, u[k]$