Optimal Power Flow over Radial Networks

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Outline

Motivation

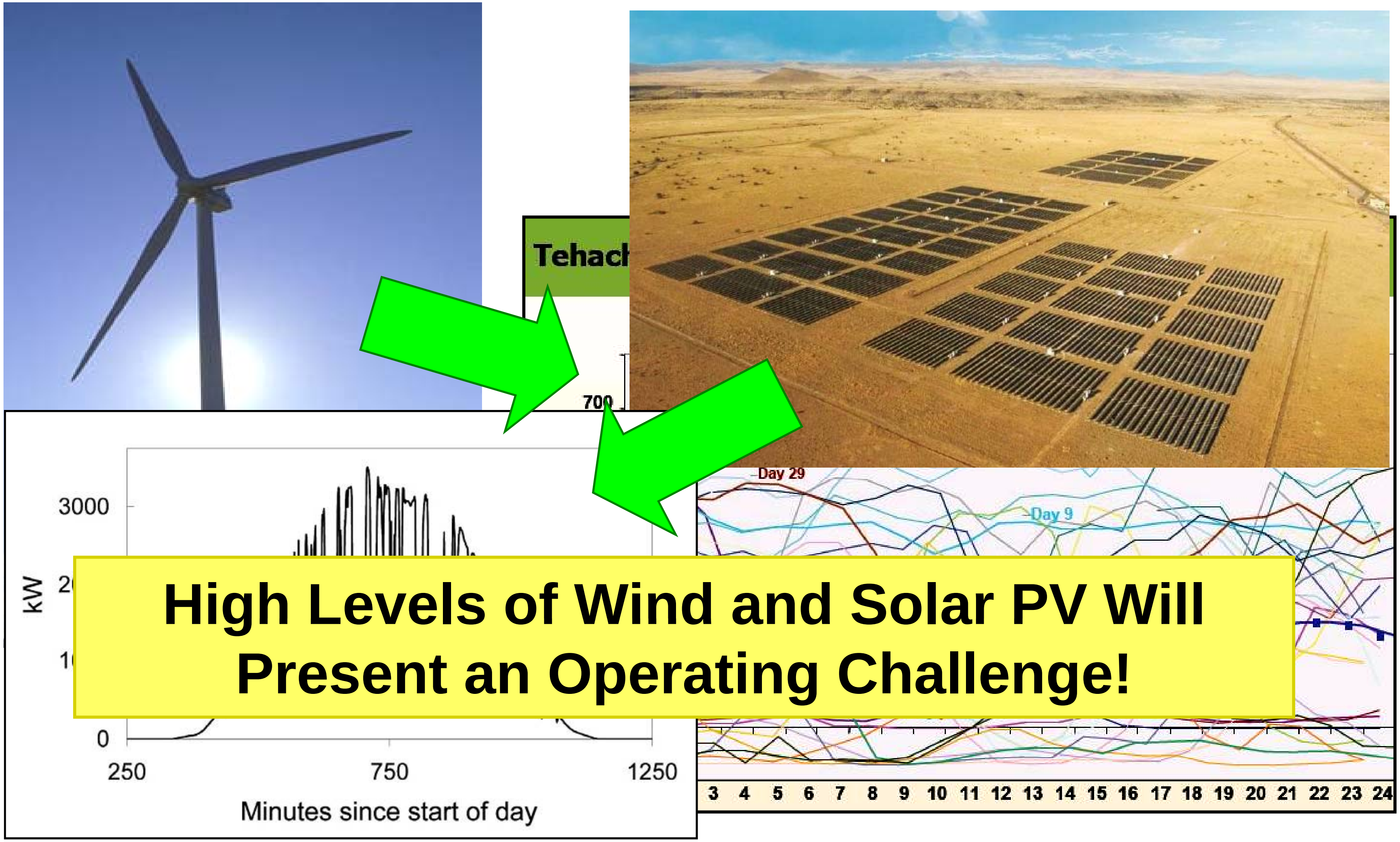
Semidefinite relaxation

- Bus injection model

Conic relaxation

- Branch flow model

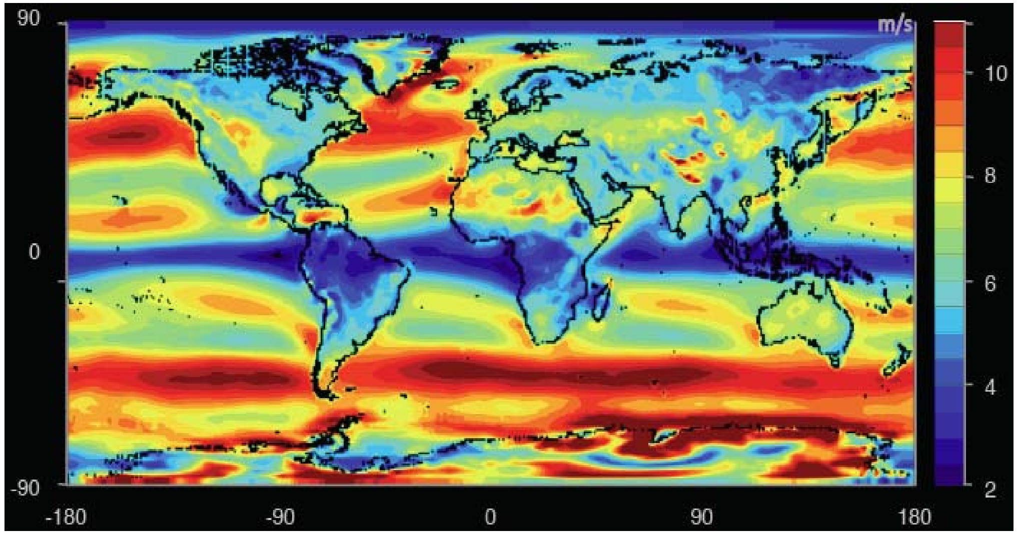
Challenge: uncertainty mgt



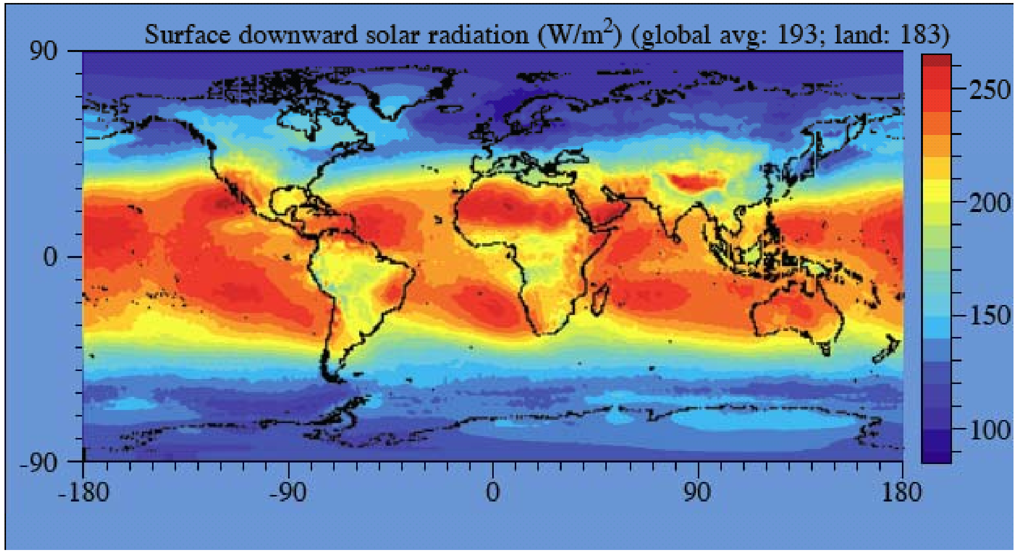
Source: Rosa Yang, EPRI

Optimal power flow (OPF)

- OPF is solved routinely to determine
  - How much power to generate where
  - Market operation & pricing
  - Parameter setting, e.g. taps, VARs
- Non-convex and hard to solve
  - Huge literature since 1962
  - In practice, operators often solve linearized model and verify using AC power flow model
- Core of many problems
  - OPF, LMP, Volt/VAR, DR, EV, planning ...



Wind power over land (exc. Antartica)
70 – 170 TW



Solar power over land
340 TW

<u>Worldwide</u>

energy demand:
16 TW

electricity demand:
2.2 TW

wind capacity (2009):
159 GW

grid-tied PV capacity (2009):
21 GW

Source: Renewable Energy
Global Status Report, 2010
Source: M. Jacobson, 2011

Implications

Current control paradigm works well today

- Low uncertainty, few active assets to control
- Centralized, open-loop, human-in-loop, worst-case preventive
- Schedule supplies to match loads

Future needs

- Fast computation to cope with rapid, random, large fluctuations in supply, demand, voltage, freq
- Simple algorithms to scale to large networks of active DER
- Real-time data for adaptive control

Implications

Must close the loop

- Real-time feedback control, risk-limiting
- Driven by uncertainty of renewables

Must be scalable

- Distributed & decentralized optimization
- Orders of magnitude more endpoints that can generate, compute, communicate, actuate

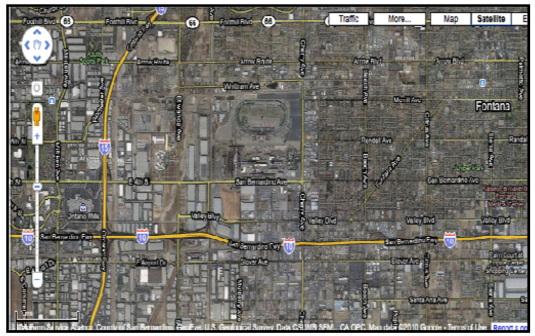
Control and optimization framework

- Theoretical foundation for a holistic framework that integrates engineering + economics
- Systematic algorithm design, understandable global behavior
- Clarify ideas, explore structures, suggest direction

Application: Volt/VAR control

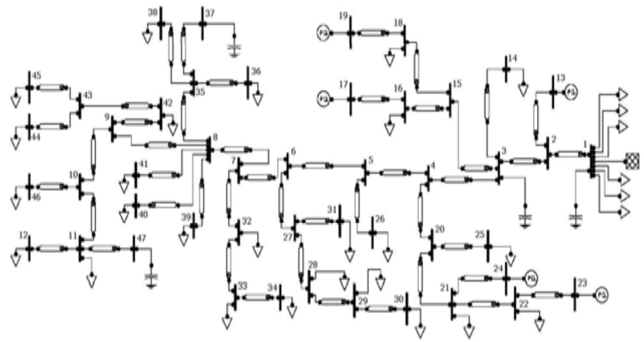
Motivation

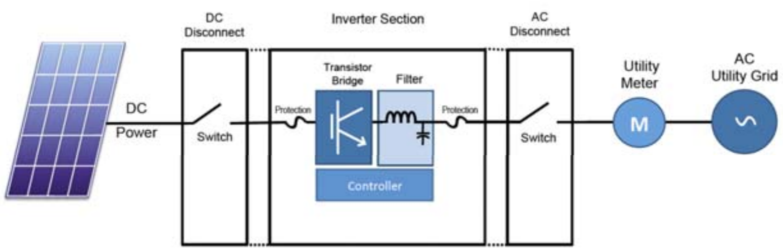
- Static capacitor control cannot cope with rapid random fluctuations of PVs on distr circuits



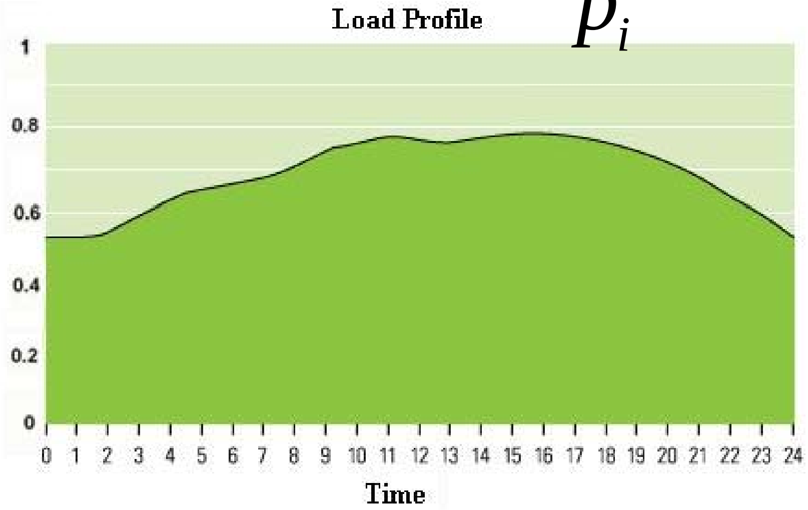
Inverter control

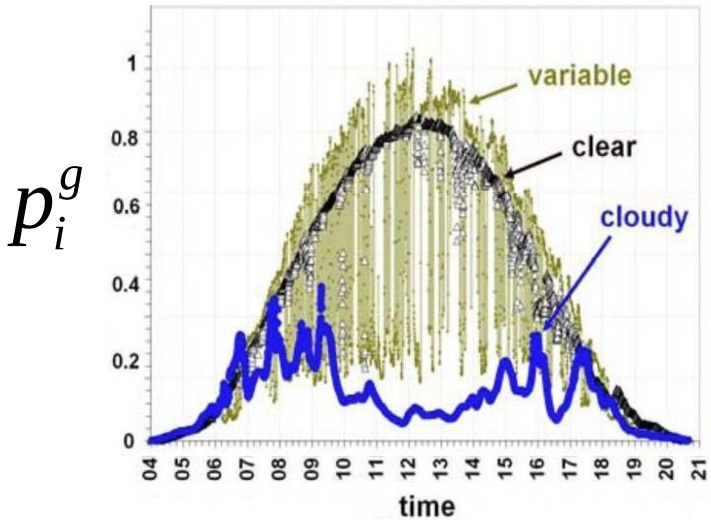
- Much faster & more frequent
- IEEE 1547 does not optimize VAR currently (unity PF)

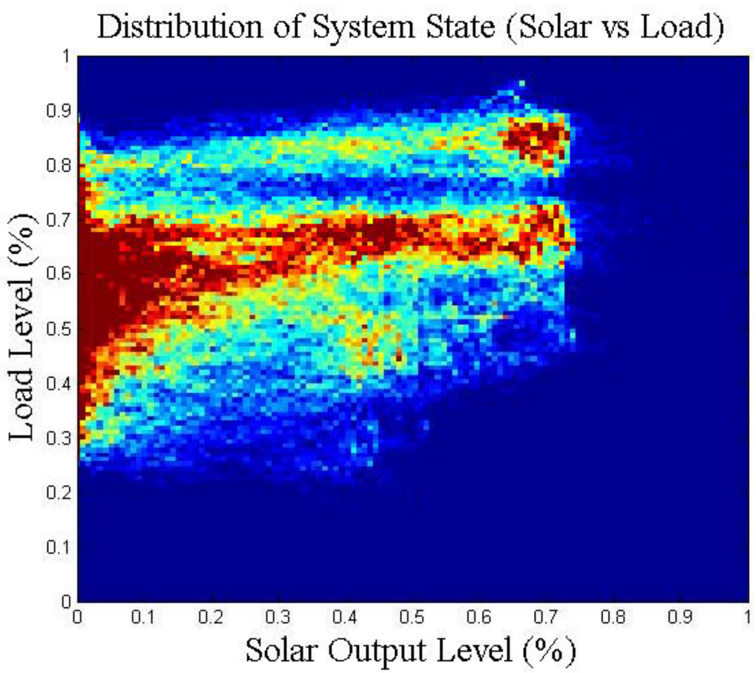




Load and Solar Variation



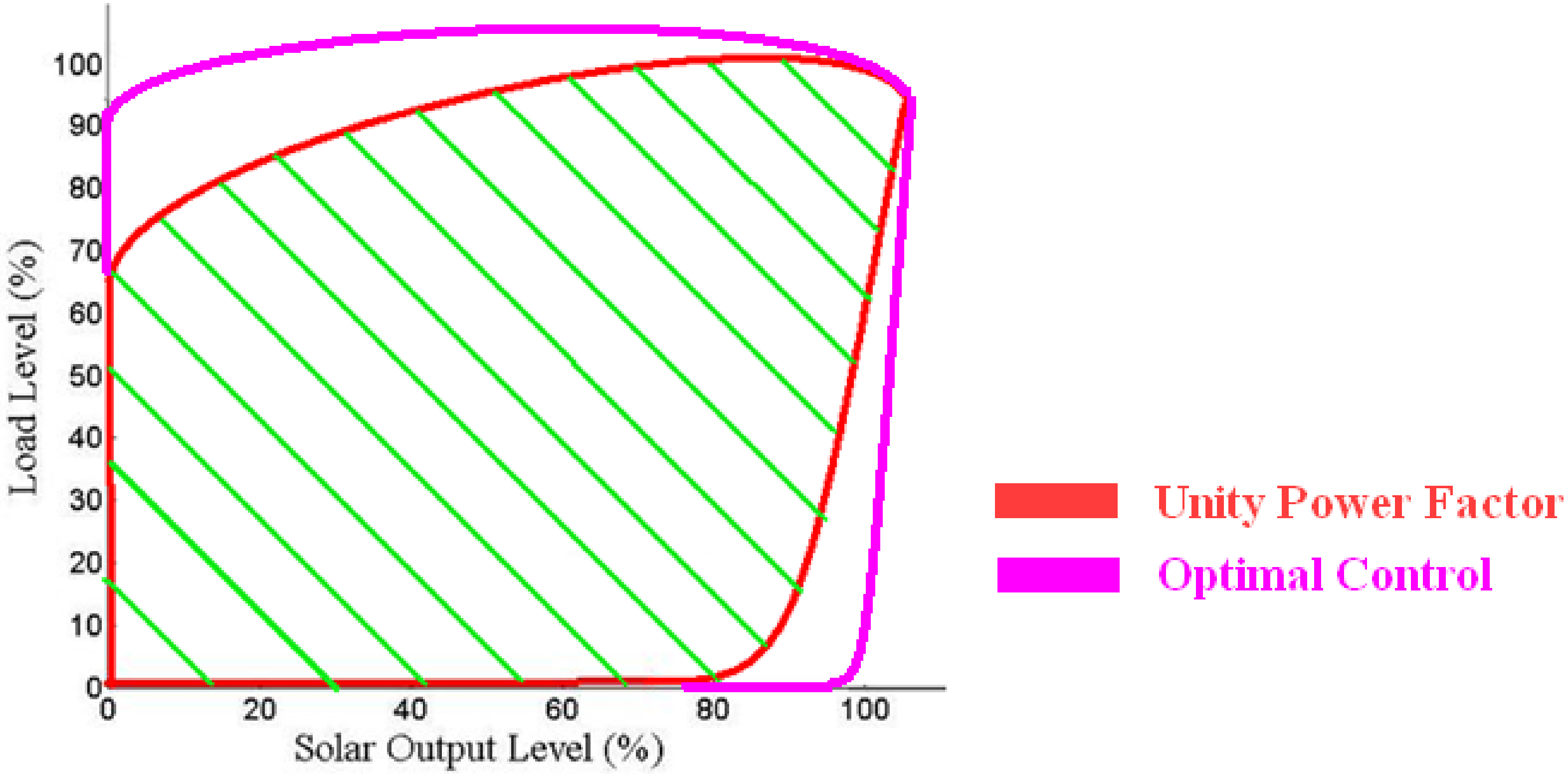




Empirical distribution
of (load, solar) for Calabash

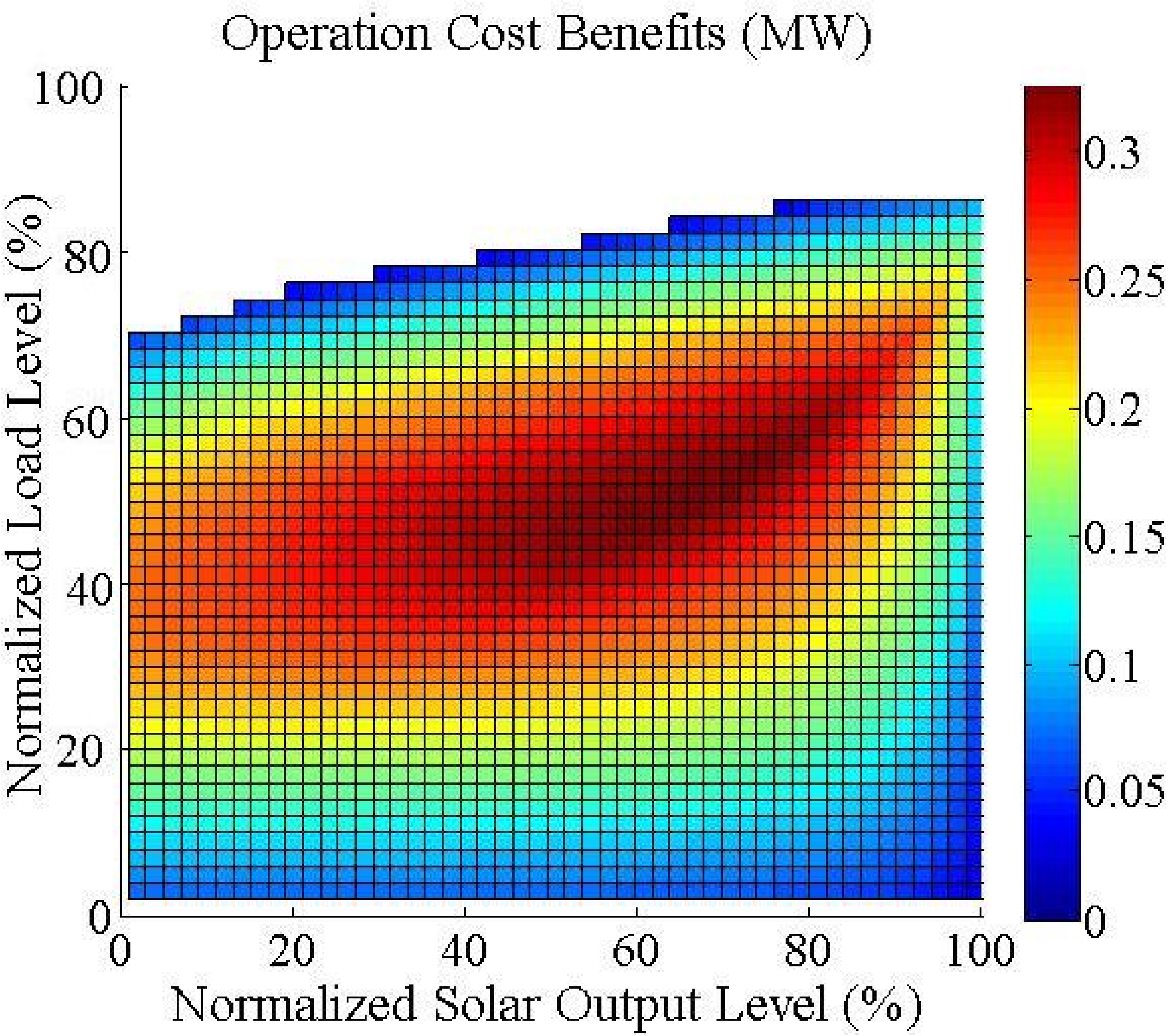
Improved reliability

$\left(p_i^g, p_i^c\right)$ for which problem is feasible



Implication: reduced likelihood of violating voltage limits or VAR flow constraints

Energy savings



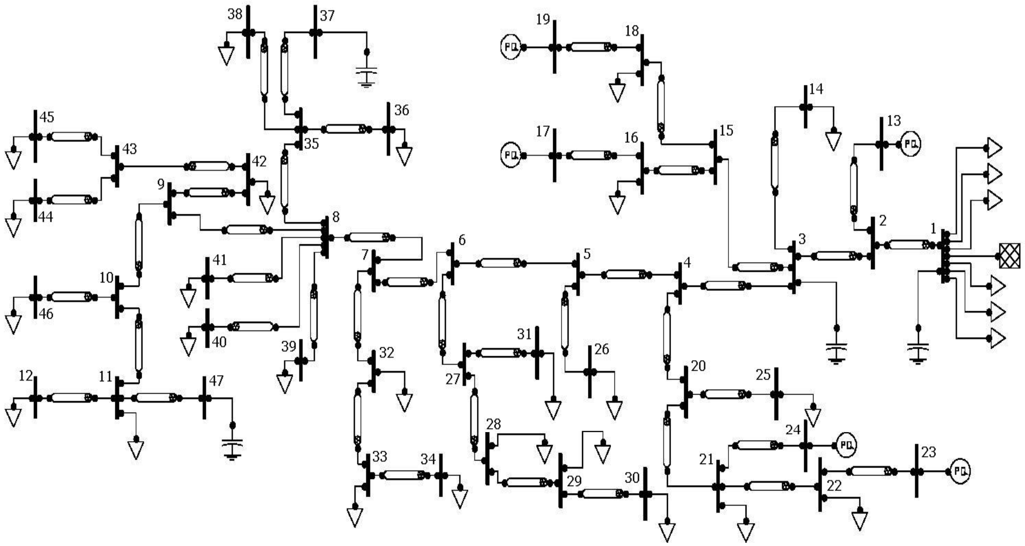
Summary

RESULTS FOR SOME VOLTAGE TOLERANCE THRESHOLDS

Voltage Drop Tolerance(pu)	Annual Hours Saved Spending Outside Feasibility Region	Average Power Saving (%)
0.03	842.9	3.93%
0.04	160.7	3.67%
0.05	14.5	3.62%

- More reliable operation
- Energy savings

Key message



Radial networks computationally simple

- Exploit tree graph & convex relaxation
- Real-time scalable control promising

Outline

Motivation

Semidefinite relaxation

- Bus injection model

Conic relaxation

- Branch flow model

Optimal power flow (OPF)

Problem formulation

- Carpentier 1962

Computational techniques:

- Dommel & Tinney 1968
- Surveys: Huneault et al 1991, Momoh et al 2001, Pandya et al 2008

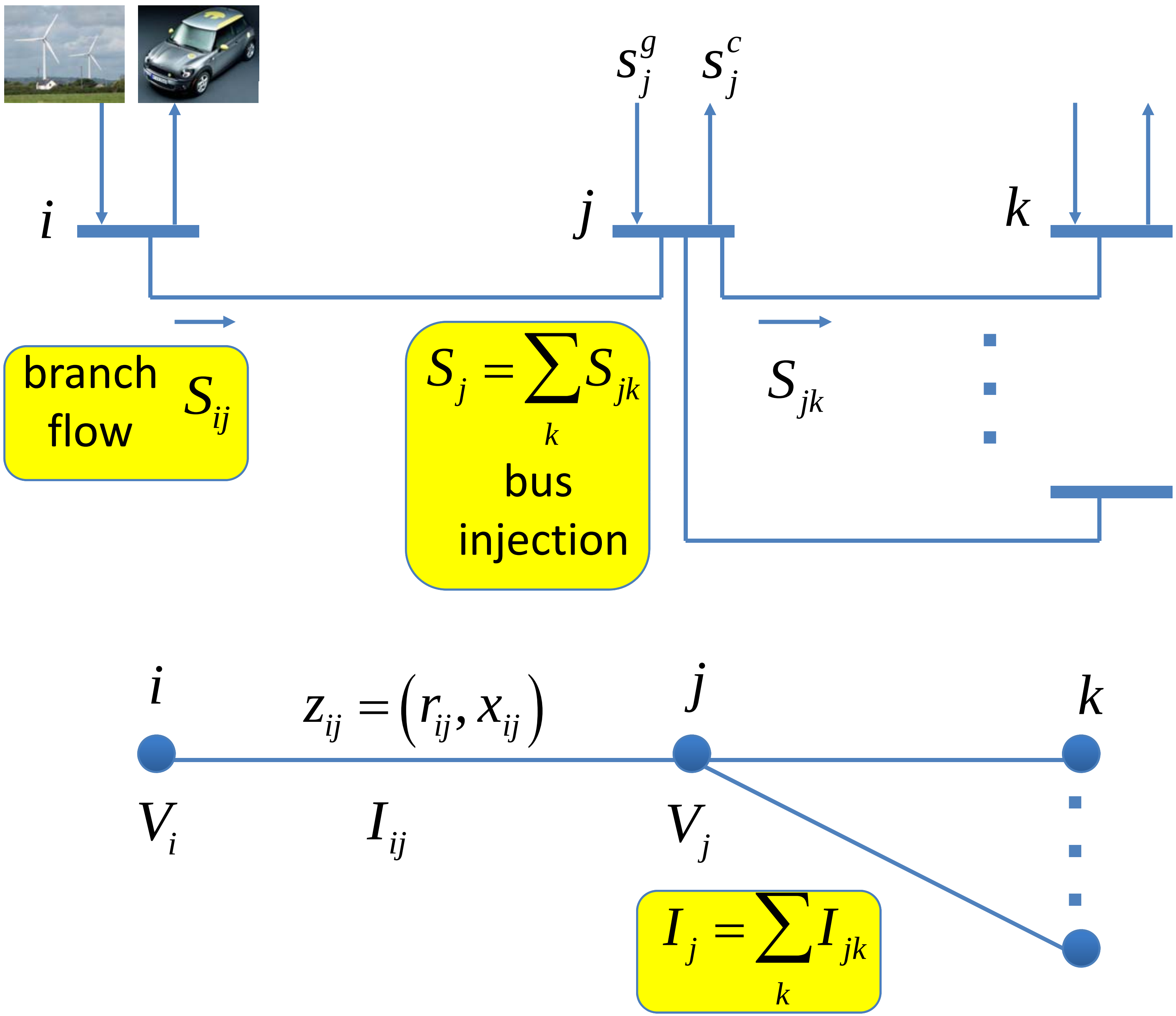
SDP formulation (bus injection model):

- Bai et al 2008, 2009, Lavaei et al 2010
- Bose et al 2011, Sojoudi et al 2011, Zhang et al 2011
- Lesieutre et al 2011

Branch flow model

- Baran & Wu 1989, Chiang & Baran 1990, Farivar et al 2011

Models



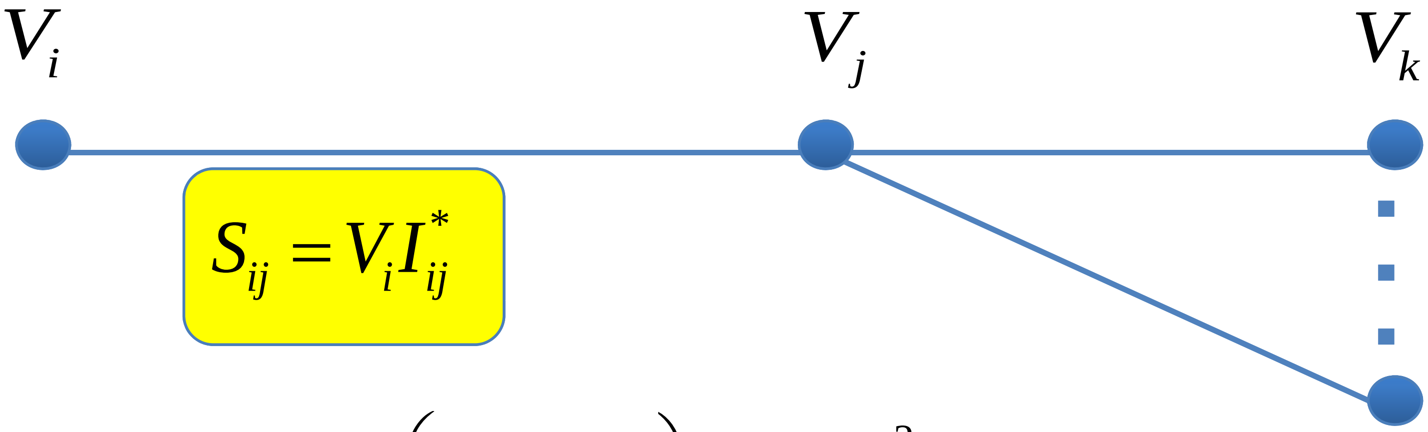
s_j^g s_j^c
i j k
branch flow S_{ij}
$S_j = \sum_k S_{jk}$
bus injection
S_{jk}
i j k
$z_{ij} = (r_{ij}, x_{ij})$
V_i I_{ij} V_j
$I_j = \sum_k I_{jk}$

Models: Kirchoff's law

$$S_i = \sum_j S_{ij} = V_i I_i^*$$

linear relation:

$$I = YV$$



$$S_{ij} = V_i I_{ij}^*$$

$$S_{ij} = V_i \left(\frac{V_i^* - V_j^*}{z_{ij}^*} \right) = \frac{|V_i|^2}{z_{ij}^*} - \frac{V_i V_j^*}{z_{ij}^*}$$

Bus injection model

Nodes *i* and *j* are linked with an admittance $y_{ij} = g_{ij} - \mathbf{i} b_{ij}$

$$Y_{ij} = \begin{cases} y_{ii} + \sum_{j \sim i} y_{ij}, & \text{if } i = j \\ -y_{ij}, & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

- Kirchhoff's Law: $I = YV$

Classical OPF

min $\sum_{k \in G} f_k\left(P_k^g\right)$ ← Generation cost

over $\left(P_k^g, Q_k^g,\ V_k\right)$

subject to $\underline{P}_k^g \le P_k^g \le \overline{P}_k^g$

$\underline{Q}_k^g \le Q_k^g \le \overline{Q}_k^g$

Generation power constraints

$\underline{V}_k \le |V_k| \le \overline{V}_k$ Voltage magnitude constraints

KVL/KCL

power balance ← nonconvexity

Classical OPF

In terms of V:

$$P_k = \text{tr}\, \Phi_k VV^*$$

$$Q_k = \text{tr}\, \Psi_k VV^*$$

$$\Phi_k := \left(\frac{Y_k^* + Y_k}{2} \right)$$

$$\Psi_k := \left(\frac{Y_k^* - Y_k}{2\mathbf{i}} \right)$$

$$\min \quad \sum_{k \in G} \text{tr}\, M_k VV^*$$

$$\text{over} \quad V$$

$$\text{s.t.} \quad \underline{P}_k^g - P_k^d \le \text{tr}\, \Phi_k VV^* \le \overline{P}_k^g - P_k^d$$

$$\underline{Q}_k^g - Q_k^d \le \text{tr}\, \Psi_k VV^* \le \overline{Q}_k^g - Q_k^d$$

$$\underline{V}_k^2 \le \text{tr}\, J_k VV^* \le \overline{V}_k^2$$

Key observation [Bai et al 2008]:
OPF = rank constrained SDP

Classical OPF

$$\min \quad \sum_{k \in G} \operatorname{tr} M_k W$$

$$\text{over} \quad W \;\; \text{positive semidefinite matrix}$$

$$\text{s.t.} \quad \underline{P}_k \leq \operatorname{tr} \Phi_k W \leq \overline{P}_k$$

$$\underline{Q}_k \leq \operatorname{tr} \Psi_k W \leq \overline{Q}_k$$

$$\underline{V}_k^2 \leq \operatorname{tr} J_k W \leq \overline{V}_k^2$$

$$W \geq 0, \quad \cancel{\operatorname{rank} W = 1}$$

convex relaxation: SDP

Semi-definite relaxation

Non-convex QCQP

↓

Rank-constrained SDP

↓

Relax the rank constraint and solve the SDP

↓

Does the optimal solution satisfy the rank-constraint?

yes → We are done!

no → Solution may not be meaningful

SDP relaxation of OPF

$$\min \quad \sum_{k \in G} \operatorname{tr} M_k W$$

over $\quad W$ positive semidefinite matrix

$$\text{s.t.} \quad \underline{P}_k \leq \operatorname{tr} \Phi_k W \leq \overline{P}_k \qquad \underline{\lambda}_k, \overline{\lambda}_k$$

$$\underline{Q}_k \leq \operatorname{tr} \Psi_k W \leq \overline{Q}_k \qquad \underline{\mu}_k, \overline{\mu}_k$$

$$\underline{V}_k^2 \leq \operatorname{tr} J_k W \leq \overline{V}_k^2 \qquad \underline{\gamma}_k, \overline{\gamma}_k$$

$$W \geq 0$$

Lagrange multipliers

$$A(\lambda_k, \mu_k, \gamma_k) := \sum_{k \in G} M_k + \sum_k (\lambda_k \Phi_k + \mu_k \Psi_k + \gamma_k J_k)$$

Sufficient condition

Theorem

If A^{opt} has rank n-1 then

- W^{opt} has rank 1, SDP relaxation is exact
- Duality gap is zero
- A globally optimal V^{opt} can be recovered

IEEE test systems (essentially) satisfy the condition!

J. Lavaei and S. H. Low: Zero duality gap in optimal power flow problem. Allerton 2010, TPS 2011

OPF over radial networks

Suppose

- tree (radial) network
- no lower bounds on power injections

Theorem

A^{opt} always has rank *n-1*

- W^{opt} always has rank 1 (exact relaxation)
- OPF always has zero V^{opt} duality gap
- Globally optimal solvable efficiently

S. Bose, D. Gayme, S. H. Low and M. Chandy, OPF over tree networks. Allerton 2011

OPF over radial networks

Suppose

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Also: B. Zhang and D. Tse, Allerton 2011
S. Sojoudi and J. Lavaei, submitted 2011

QCQP over tree

QCQP (C, C_k)

$$\begin{aligned} &\min && x^* C x && \\ &\text{over} && x \in \mathbf{C}^n && \\ &\text{s.t.} && x^* C_k x \ \le \ b_k && k \in K \end{aligned}$$

graph of QCQP

$$G(C, C_k) \ \text{ has edge } (i, j) \quad \Leftrightarrow$$

$$C_{ij} \neq 0 \ \text{ or } \ [C_k]_{ij} \neq 0 \ \text{ for some } k$$

QCQP over tree

$$G(C, C_k) \ \text{ is a tree}$$

QCQP over tree

QCQP (C, C_k)

$$\begin{array}{lll} \min & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \ \le \ b_k & k \in K \end{array}$$

Semidefinite relaxation

$$\begin{array}{lll} \min & \operatorname{tr} CW & \\ \text{over} & W \ge 0 & \\ \text{s. t.} & \operatorname{tr} C_k W \le b_k & k \in K \end{array}$$

QCQP over tree

QCQP (C, C_k)

$$\begin{array}{lll} \min & x^* C x & \\ \text{over} & x \in \mathbf{C}^n & \\ \text{s.t.} & x^* C_k x \ \le \ b_k & k \in K \end{array}$$

Key assumption

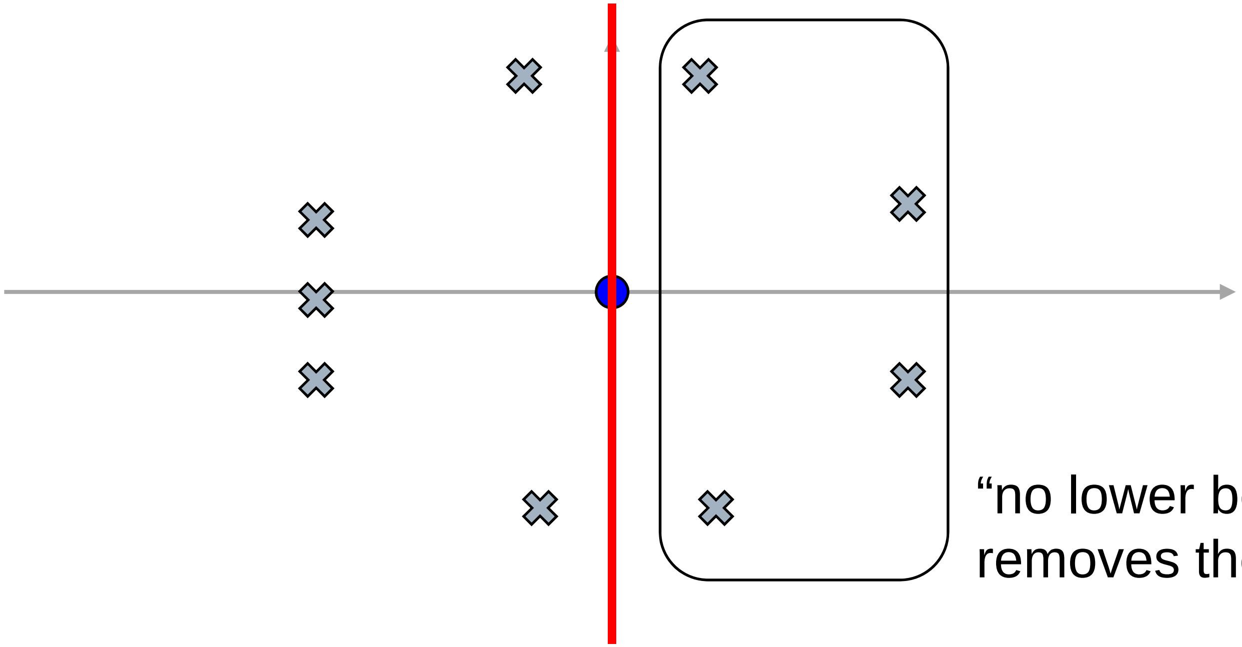
$$(i,j) \in G(C, C_k): \quad 0 \notin \text{int conv hull} \left(C_{ij}, [C_k]_{ij}, \ \forall k \right)$$

Theorem

Semidefinite relaxation is exact for QCQP over tree

S. Bose, D. Gayme, S. H. Low and M. Chandy, submitted March 2012

OPF over radial networks



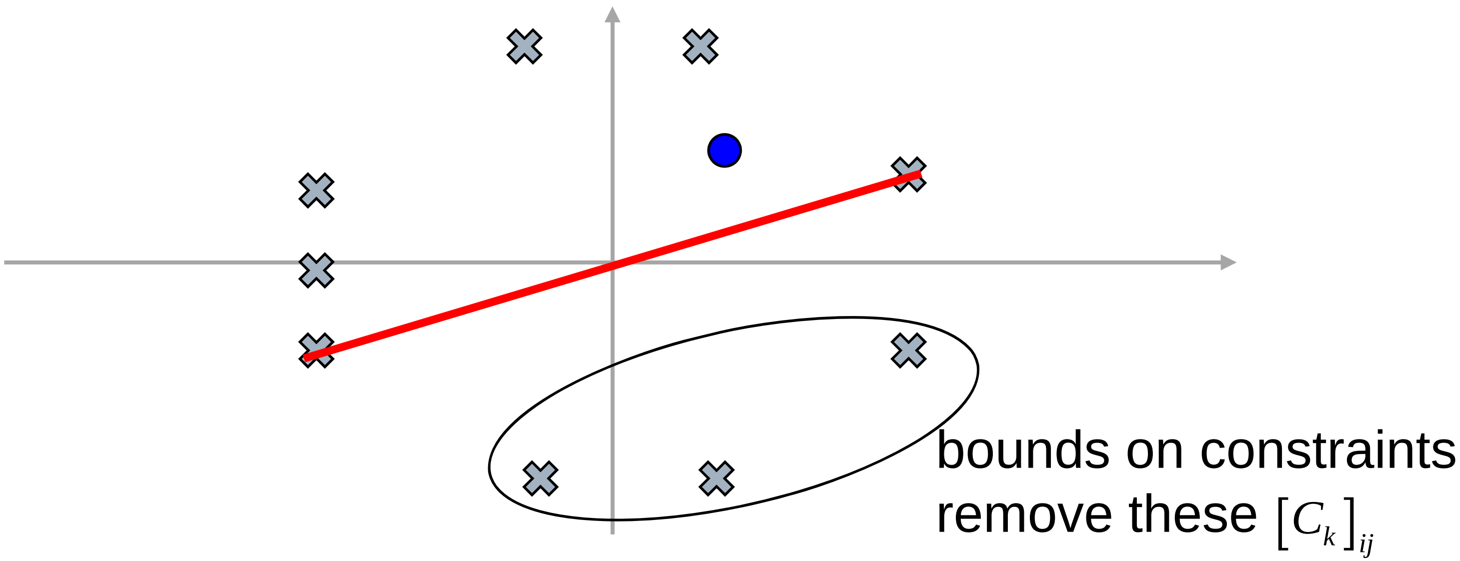
"no lower bounds" removes these $[C_k]_{ij}$

Theorem

A^{opt} always has rank *n-1*

- W^{opt} always has rank 1 (exact relaxation)
- OPF always has zero duality gap
- Globally optimal ... solvable efficiently

OPF over radial networks



Theorem

A^{opt} always has rank *n-1*

- ☐ W^{opt} always has rank 1 (exact relaxation)
- ☐ OPF always has zero V^{opt} duality gap
- ☐ Globally optimal solvable efficiently

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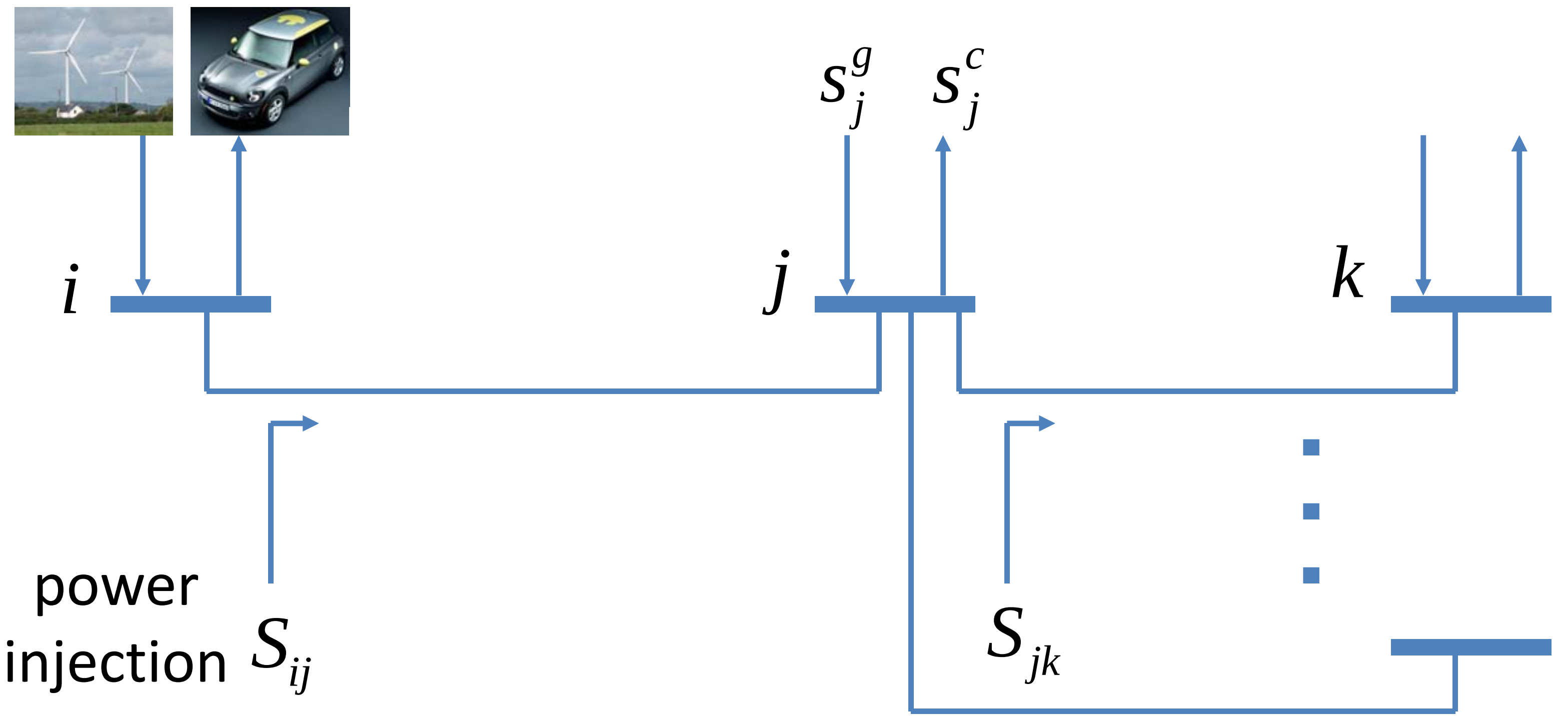
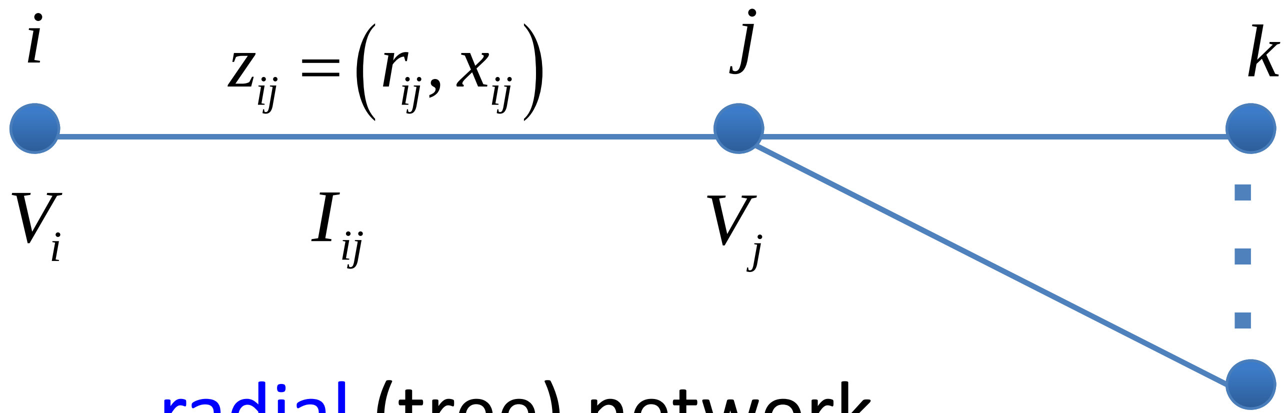
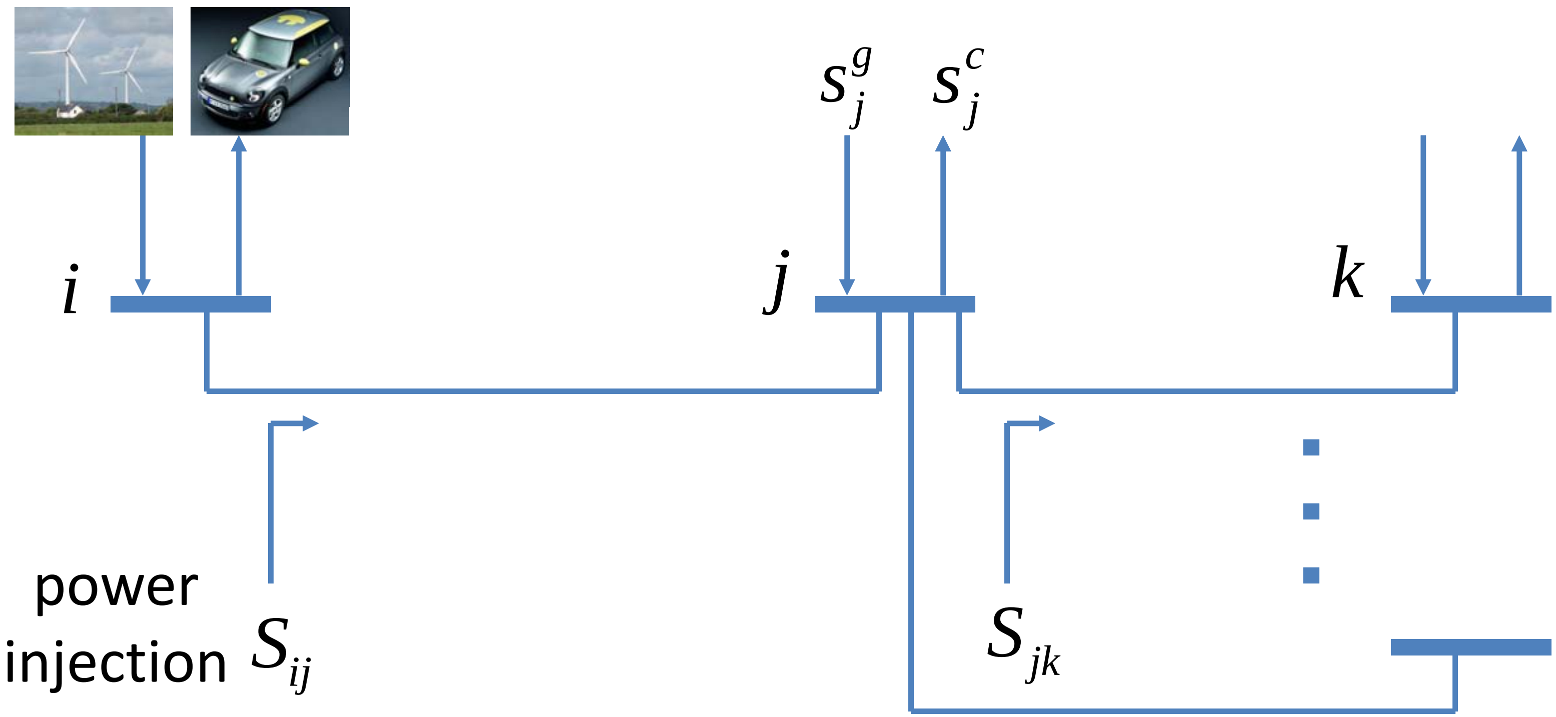
Model





radial (tree) network

Model

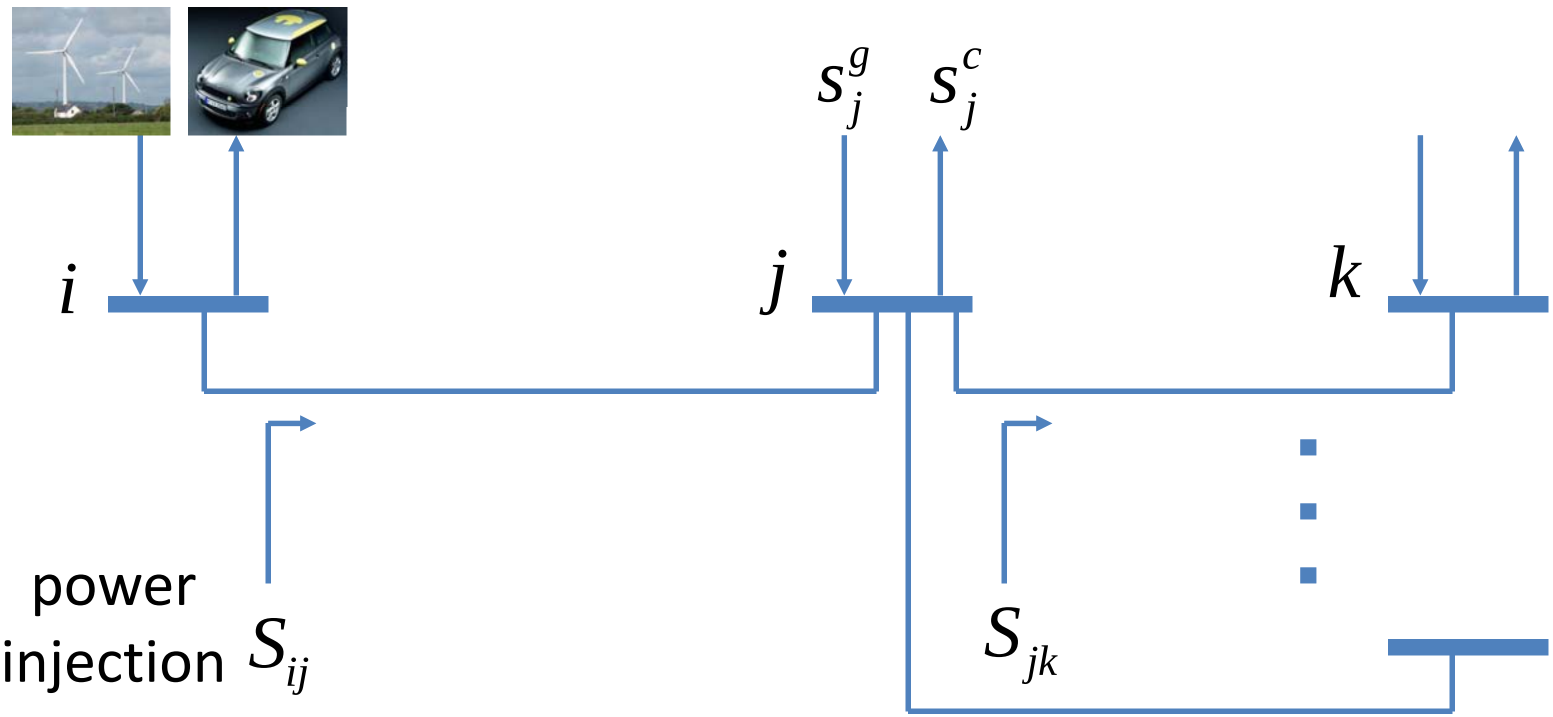


Kirchoff's Law: $$S_{ij} = \sum_{k:j \sim k} S_{jk} + z_{ij} \left| I_{ij} \right|^2 + s_j^c - s_j^g$$

line loss

load - gen

Model



Kirchoff's Law: $$S_{ij} = \sum_{k:j\sim k} S_{jk} + z_{ij}\left|I_{ij}\right|^2 + s_j^c - s_j^g$$

Ohm's Law: $$V_j = V_i - z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^*$$

OPF

$$\min \sum_{i\sim j} r_{ij} l_{ij} + \sum_i \alpha_i v_i$$

real power loss

CVR (conservation voltage reduction)

$$l_{ij} := |I_{ij}|^2$$

$$v_i := |V_i|^2$$

OPF using branch flow model

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_i \alpha_i v_i$$

$$\text{over} \quad (S, I, V, s^g, s^c)$$

$$\text{s. t.} \quad \underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c$$

$$\underline{v}_i \le v_i \le \overline{v}_i$$

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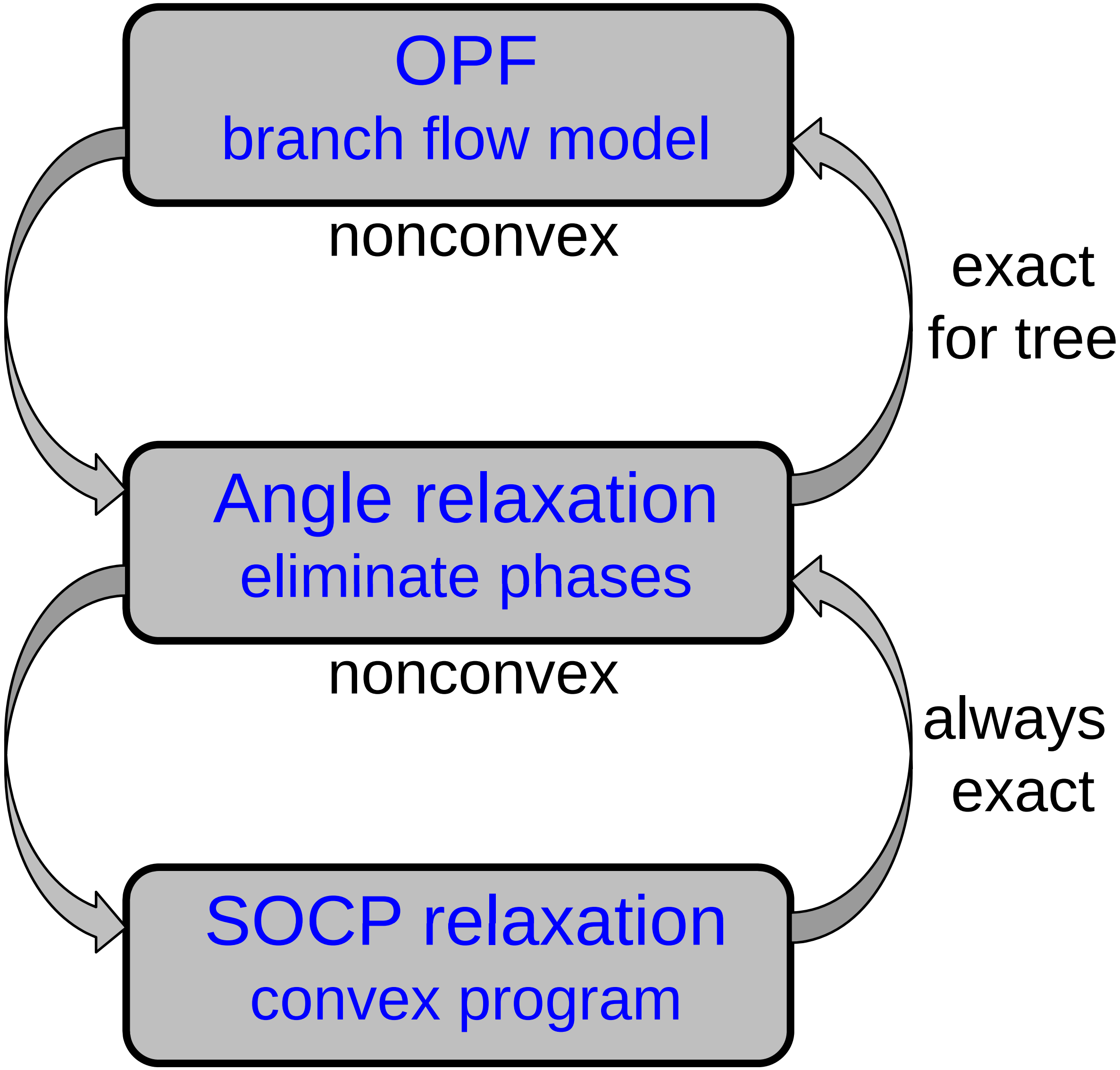
Solution strategy



1. Angle relaxation

Angles of I_i , V_i eliminated !
Points relaxed to circles

demands

$$P_{ij} = \sum_{k:j\sim k} P_{jk} + r_{ij}\left|I_{ij}\right|^2 + p_j^c - p_j^g$$

$$Q_{ij} = \sum_{k:j\sim k} Q_{jk} + x_{ij}\left|I_{ij}\right|^2 + q_j^c - q_j^g$$

$$\left|V_i\right|^2 = \left|V_j\right|^2 + 2\left(r_{ij}P_{ij} + x_{ij}Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right)\left|I_{ij}\right|^2$$

$$\left|I_{ij}\right|^2 = \left(\frac{P_{ij}^2 + Q_{ij}^2}{\left|V_i\right|^2}\right)$$

Baran and Wu 1989
for **radial** networks

1. Angle relaxation

Angles of I_i , V_i eliminated !
Points relaxed to circles

$$P_{ij} = \sum_{k:j\sim k} P_{jk} + r_{ij} \left| I_{ij} \right|^2 + p_j^c - p_j^g$$ ← generation

$$Q_{ij} = \sum_{k:j\sim k} Q_{jk} + x_{ij} \left| I_{ij} \right|^2 + q_j^c - q_j^g$$ ← VAR control

$$\left| V_i \right|^2 = \left| V_j \right|^2 + 2\left(r_{ij} P_{ij} + x_{ij} Q_{ij} \right) - \left(r_{ij}^2 + x_{ij}^2 \right) \left| I_{ij} \right|^2$$

$$\left| I_{ij} \right|^2 = \left(\frac{P_{ij}^2 + Q_{ij}^2}{\left| V_i \right|^2} \right)$$

$$\underline{s}_i^g \le s_i^g \le \overline{s}_i^g \qquad \underline{s}_i \le s_i^c$$

$$\underline{v}_i \le v_i \le \overline{v}_i$$

1. Angle relaxation

$$l_{ij} := |I_{ij}|^2$$

$$v_i := |V_i|^2$$

- Linear objective
- Linear constraints
- Quadratic equality

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_i \alpha_i v_i$$

$$\text{over} \quad (S, l, v, s^g, s^c)$$

$$\text{s. t.} \quad P_{ij} = \sum_{k:j\sim k} P_{jk} + r_{ij} l_{ij} + p_j^c - p_j^g$$

$$Q_{ij} = \sum_{k:j\sim k} Q_{jk} + x_{ij} l_{ij} + q_j^c - q_j^g$$

$$v_i = v_j + 2\left(r_{ij} P_{ij} + x_{ij} Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right) l_{ij}$$

$$l_{ij} = \left(\frac{P_{ij}^2 + Q_{ij}^2}{v_i}\right), \qquad \underline{s}_i \le s_i^g \le \overline{s}_i$$

$$\underline{v}_i \le v_i \le \overline{v}_i, \qquad \underline{s}_i \le \overline{s}_i^c$$

2. SOCP relaxation

$$\min \quad \sum_{i\sim j} r_{ij} l_{ij} + \sum_i \alpha_i v_i$$

$$\text{over} \quad (S, l, v, s^g, s^c)$$

$$\text{s. t.} \quad P_{ij} = \sum_{k: j\sim k} P_{jk} + r_{ij} l_{ij} + p_j^c - p_j^g$$

$$Q_{ij} = \sum_{k: j\sim k} Q_{jk} + x_{ij} l_{ij} + q_j^c - q_j^g$$

$$v_i = v_j + 2\left(r_{ij} P_{ij} + x_{ij} Q_{ij}\right) - \left(r_{ij}^2 + x_{ij}^2\right) l_{ij}$$

Quadratic inequality

$$l_{ij} \geq \left(\frac{P_{ij}^2 + Q_{ij}^2}{v_i} \right) \qquad \underline{s}_i \leq s_i^g \leq \overline{s}_i$$

$$\underline{v}_i \leq v_i \leq \overline{v}_i, \qquad \underline{s}_i \leq s_i^c$$

OPF over radial networks

Theorem

Both relaxation steps are exact

- SOCP relaxation is (convex and) exact
- Phase angles can be uniquely determined

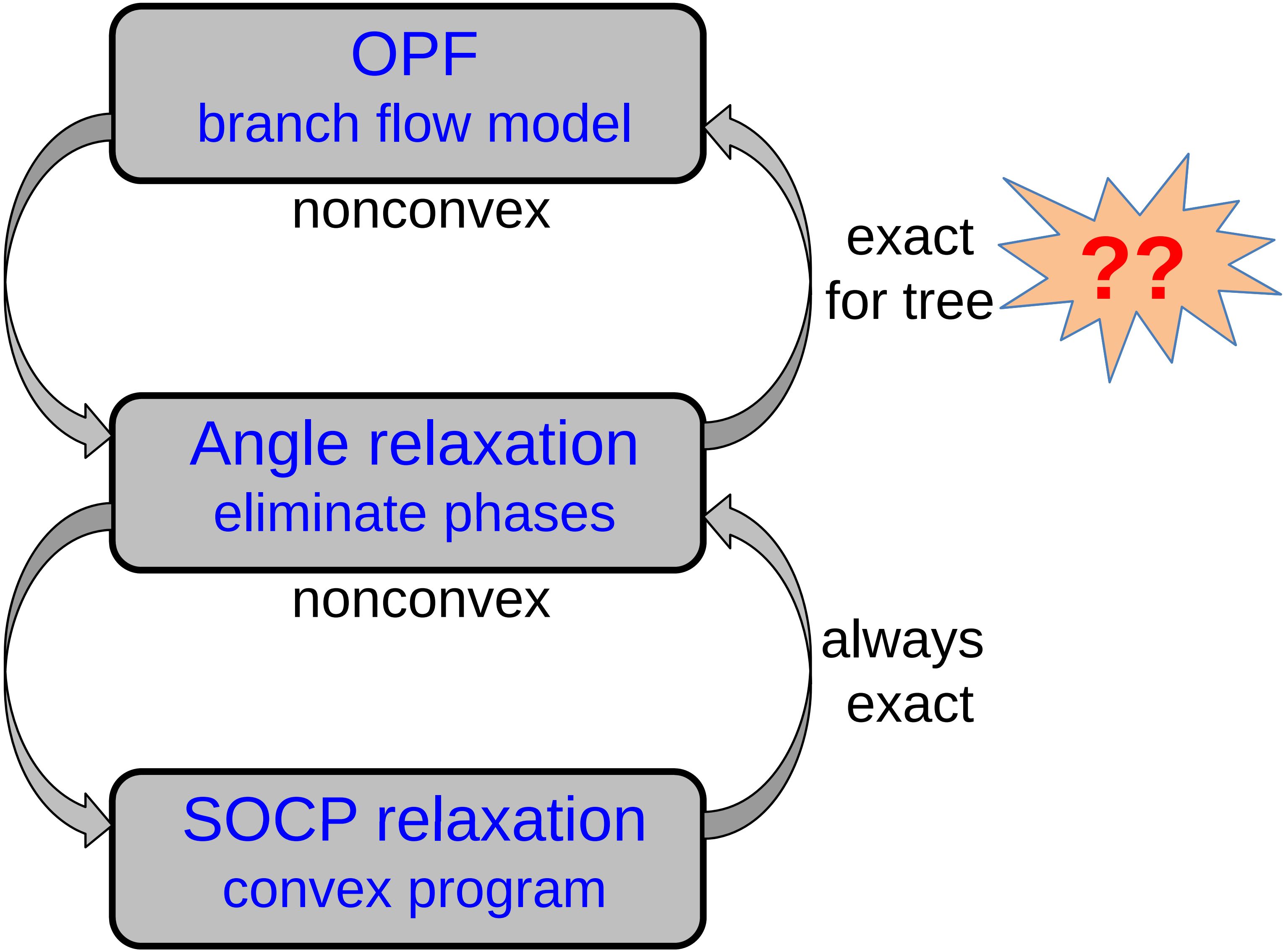
Original OPF problem has zero duality gap

M. Farivar, C. Clarke, S. H. Low and M. Chandy, Inverter VAR control for distribution systems with renewables. SmartGridComm 2011

What about mesh networks ??

M. Farivar and S. H. Low, submitted March 2012

Solution strategy



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Ohm's Law: $V_j = V_i - z_{ij} I_{ij}$

$$S_{ij} = V_i I_{ij}^*$$

Convexification of mesh networks

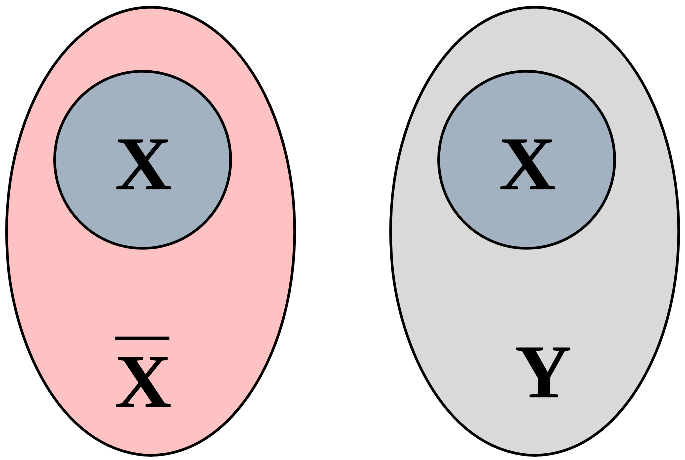
OPF $\min_{x,s} \; f\left(\hat{h}(x)\right)$ s.t. $x \in \mathbf{X}, \;\; s \in \mathbf{S}$

OPF-ar $\min_{x,s} \; f\left(\hat{h}(x)\right)$ s.t. $x \in \mathbf{Y}, \;\; s \in \mathbf{S}$

OPF-ps $\min_{x,s,\phi} \; f\left(\hat{h}(x)\right)$ s.t. $x \in \overline{\mathbf{X}}, \;\; s \in \mathbf{S}$

Theorem

- $\overline{\mathbf{X}} = \mathbf{Y}$
- Need phase shifters only outside spanning tree



Convexification of mesh networks

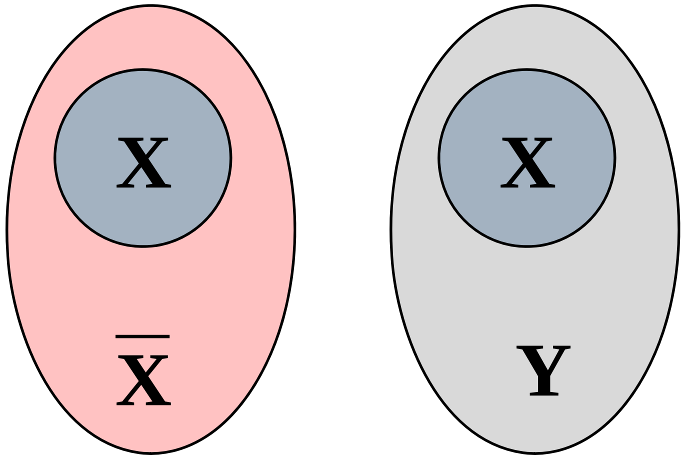
OPF $\quad \min_{x,s} \; f\left(\hat{h}(x)\right) \quad \text{s.t.} \quad x \in \mathbf{X}$

OPF-ar $\quad \min_{x,s} \; f\left(\hat{h}(x)\right) \quad \text{s.t.} \quad x \in \mathbf{Y}$

OPF-ps $\quad \min_{x,s,\phi} \; f\left(\hat{h}(x)\right) \quad \text{s.t.} \quad x \in \overline{\mathbf{X}}$

Theorem

- $\overline{\mathbf{X}} = \mathbf{Y}$
- Need phase shifters only outside spanning tree



Key message

Radial networks computationally simple

- Exploit tree graph & convex relaxation
- Real-time scalable control promising

Mesh networks can be convexified

- Design for simplicity
- Need few phase shifters (sparse topology)