18-791 Lecture #17
INTRODUCTION TO THE FAST FOURIER TRANSFORM ALGORITHM

Richard M. Stern

Department of Electrical and Computer Engineering
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

Phone: +1 (412) 268-2535
FAX: +1 (412) 268-3890
rms@cs.cmu.edu
http://www.ece.cmu.edu/~rms

October 24, 2005

Introduction

- **Today we will begin our discussion of the family of algorithms known as "Fast Fourier Transforms", which have revolutionized digital signal processing**
- **What is the FFT?**
  - A collection of "tricks" that exploit the symmetry of the DFT calculation to make its execution much faster
  - Speedup increases with DFT size
- **Today - will outline the basic workings of the simplest formulation, the radix-2 decimation-in-time algorithm**
- **Thursday - will discuss some of the variations and extensions**
  - Alternate structures
  - Non-radix 2 formulations

Introduction, continued

- **Some dates:**
  - ~1880 - algorithm first described by Gauss
  - 1965 - algorithm rediscovered (not for the first time) by Cooley and Tukey
- **In 1967 (spring of my freshman year), calculation of a 8192-point DFT on the top-of-the line IBM 7094 took ….**
  - ~30 minutes using conventional techniques
  - ~5 seconds using FFTs

Measures of computational efficiency

- **Could consider**
  - Number of additions
  - Number of multiplications
  - Amount of memory required
  - Scalability and regularity
- **For the present discussion we'll focus most on number of multiplications as a measure of computational complexity**
  - More costly than additions for fixed-point processors
  - Same cost as additions for floating-point processors, but number of operations is comparable

Computational Cost of Discrete-Time Filtering

Convolution of an *N*-point input with an *M*-point unit sample response ….

- **Direct convolution:**

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  – Number of multiplies ≈ *MN*

Computational Cost of Discrete-Time Filtering

Convolution of an *N*-point input with an *M*-point unit sample response ….

- **Using transforms directly:**

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

  – Computation of *N*-point DFTs requires N^2 multiplys

  – Each convolution requires three DFTs of length *N+M-1* plus an additional *N+M-1* complex multiplys or

$$3(N+M-1)^2 + (N+M-1)$$

  – For $N >> M$, for example, the computation is $O(N^2)$

Computational Cost of Discrete-Time Filtering

Convolution of an *N*-point input with an *M*-point unit sample response ….

- **Using overlap-add with sections of length *L*:**
  - *N/L* sections, 2 DFTs per section of size *L+M-1*, plus additional multiplys for the DFT coefficients, plus one more DFT for $h[n]$

$$\frac{2N}{L}(L+M-1)^2+\left(\frac{N}{L}\right)(L+M-1)+(L+M-1)^2$$

  - For very large *N*, still is proportional to M^2

The Cooley-Tukey decimation-in-time algorithm

- **Consider the DFT algorithm for an integer power of 2,** $N=2^{\nu}$

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N{}^{nk}=\sum_{n=0}^{N-1}x[n]e^{-j2\pi nk/N};\ W_N=e^{-j2\pi/N}$$

- **Create separate sums for even and odd values of *n:***

$$X[k]=\sum_{n\ even}x[n]W_N{}^{nk}+\sum_{n\ odd}x[n]W_N{}^{nk}$$

- **Letting** $n=2r$ **for *n* even and** $n=2r+1$ **for *n* odd, we obtain**

$$X[k]=\sum_{r=0}^{(N/2)-1}x[2r]W_N{}^{2rk}+\sum_{r=0}^{(N/2)-1}x[2r+1]W_N{}^{(2r+1)k}$$

The Cooley-Tukey decimation in time algorithm

- **Splitting indices in time, we have obtained**

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r]W_N{}^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]W_N{}^{(2r+1)k}$$

- **But** $W_N^2 = e^{-j2\pi 2/N} = e^{-j2\pi/(N/2)} = W_{N/2}$ **and** $W_N^{2rk}W_N^k = W_N^k W_{N/2}^{rk}$

So …

$$X[k] = \underbrace{\sum_{n=0}^{(N/2)-1} x[2r]W_{N/2}^{rk}}_{} + W_N^k \underbrace{\sum_{n=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}}_{}$$

N/2*-point DFT of *x[2r] ***N/2*-point DFT of *x[2r+1]***

Savings so far …

- **We have split the DFT computation into two halves:**

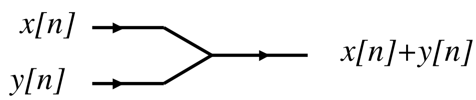
$$X[k] = \sum_{k=0}^{N-1} x[n]W_N{}^{nk}$$

$$= \sum_{n=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{n=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}$$

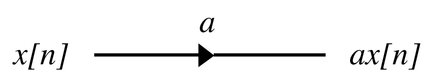
- **Have we gained anything? Consider the nominal number of multiplications for** $N = 8$
  - Original form produces $8^2 = 64$ multiplications
  - New form produces $2(4^2) + 8 = 40$ multiplications
  - So we're already ahead ….. Let's keep going!!

Signal flowgraph notation

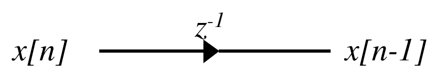
- In generalizing this formulation, it is most convenient to adopt a graphic approach …
- Signal flowgraph notation describes the three basic DSP operations:
  - Addition



  - Multiplication by a constant

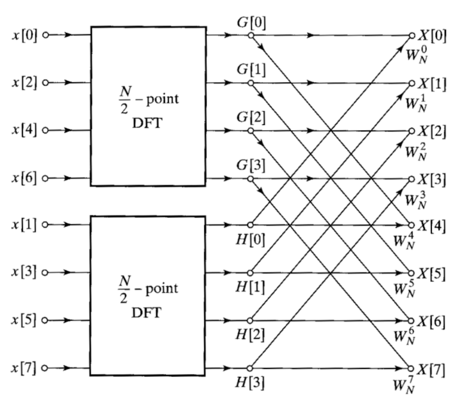


  - Delay



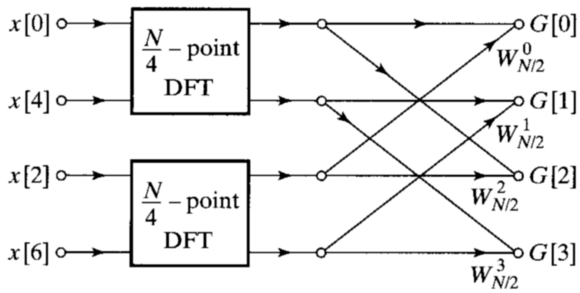
Signal flowgraph representation of 8-point DFT

- Recall that the DFT is now of the form $X[k] = G[k] + W_N^k H[k]$
- The DFT in (partial) flowgraph notation:

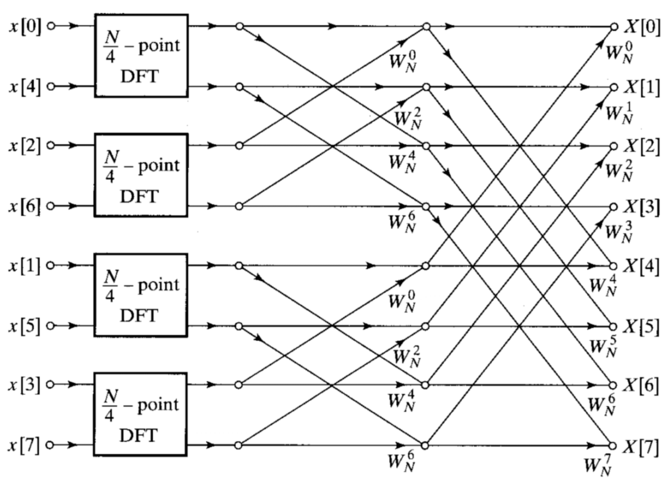


Continuing with the decomposition …

- **So why not break up into additional DFTs? Let’s take the upper 4-point DFT and break it up into two 2-point DFTs:**



The complete decomposition into 2-point DFTs



Now let's take a closer look at the 2-point DFT

- **The expression for the 2-point DFT is:**

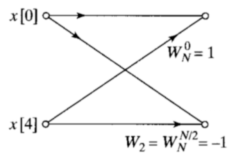
$$X[k] = \sum_{n=0}^{1} x[n]W_2^{nk} = \sum_{n=0}^{1} x[n]e^{-j2\pi nk/2}$$

- **Evaluating for $k = 0,1$ we obtain**

$$X[0] = x[0] + x[1]$$

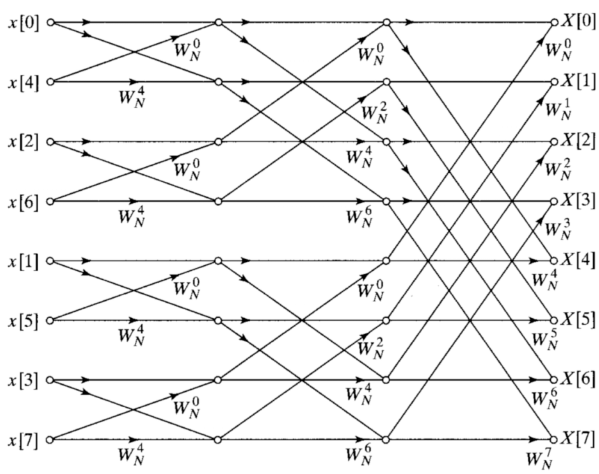
$$X[1] = x[0] + e^{-j2\pi 1/2}x[1] = x[0] - x[1]$$

which in signal flowgraph notation looks like ...

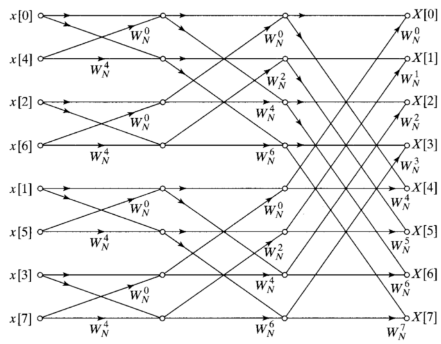


This topology is referred to as the basic butterfly

The complete 8-point decimation-in-time FFT



Number of multiplys for N-point FFTs



- **Let** $N = 2^{\nu}$ where $\nu = \log_2(N)$
- **($\log_2$(*N*) columns)(*N/2* butterflys/column)(2 mults/butterfly)**

  or ~ $N \log_2(N)$ **multiplys**

Comparing processing with and without FFTs

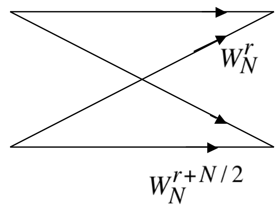
- **"Slow" DFT requires *N* mults; FFT requires *N* $\log_2$(*N*) mults**
- **Filtering using FFTs requires 3(*N* $\log_2$(*N*))+2*N* mults**
- **Let** $\alpha_1 = N \log_2(N) / N^2; \quad \alpha_2 = [3(N \log_2(N)) + N] / N^2$

N	α_1	α_2
16	.25	.8124
32	.156	.50
64	.0935	.297
128	.055	.171
256	.031	.097
1024	.0097	.0302

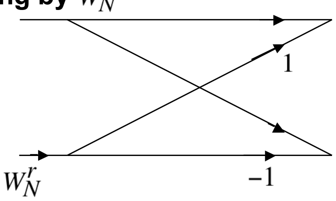
Note: 1024-point FFTs accomplish speedups of 100 for filtering, 30 for DFTs!

Additional timesavers: reducing multiplications in the basic butterfly

- As we derived it, the basic butterfly is of the form

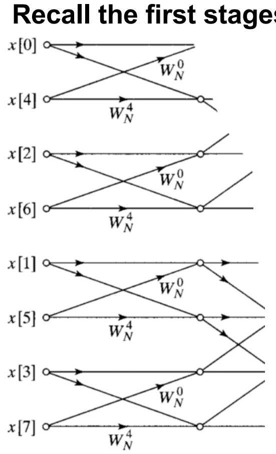


- Since $W_N^{N/2} = -1$ we can reducing computation by 2 by premultiplying by W_N^r



Bit reversal of the input

- Recall the first stages of the 8-point FFT:



Consider the binary representation of the indices of the input:

0 000
4 100
2 010
6 110
1 001
5 101
3 011
7 111

If these binary indices are time reversed, we get the binary sequence representing 0,1,2,3,4,5,6,7

Hence the indices of the FFT inputs are said to be in bit-reversed order

Some comments on bit reversal

- In the implementation of the FFT that we discussed, the input is bit reversed and the output is developed in natural order
- Some other implementations of the FFT have the input in natural order and the output bit reversed (to be described Thursday)
- In some situations it is convenient to implement filtering applications by
  - Use FFTs with input in natural order, output in bit-reversed order
  - Multiply frequency coefficients together (in bit-reversed order)
  - Use inverse FFTs with input in bit-reversed order, output in natural order
- Computing in this fashion means we never have to compute bit reversal explicitly

Summary

- We developed the structure of the basic decimation-in-time FFT
- Use of the FFT algorithm reduces the number of multiplys required to perform the DFT by a factor of more than 100 for 1024-point DFTs, with the advantage increasing with increasing DFT size
- Next time we will consider inverse FFTs, alternate forms of the FFT, and FFTs for values of DFT sizes that are not an integer power of 2