18-791 Lecture #8
INTRODUCTION TO THE *Z*-TRANSFORM

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Introduction

- **Last week we discussed the discrete-time Fourier transform (DTFT) at length**
- **This week we will begin our discussion of the Z-transform (ZT)**
  - ZT can be thought of as a generalization of the DTFT
  - ZT is more complex than DTFT (both literally and figuratively), but provides a great deal of insight into system design and behavior
- **So today we will:**
  - Define ZTs and their regions of convergence (ROC)
  - Provide insight into the relationships between frequency using ZT and DTFT relationships
  - Discuss relations between unit sample response and shape of ROC

The Discrete-Time Fourier Transform (DTFT) and the *Z*-transform (ZT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}\,d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

- The first equation aserts that we can represent any time function *x[n]* by a linear combination of complex exponentials

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

- The second equation tells us how to compute the complex weighting factors $X(e^{j\omega})$
- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n

Generalizing the frequency variable

- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n
- z^n can be thought of as a generalization of $e^{j\omega n}$
- For an arbitrary z, using polar notation we obtain $z = \rho e^{j\omega}$ so

$$z^n = \rho^n e^{j\omega n}$$

- If both ρ and ω are real, then z^n can be thought of as a complex exponential (*i.e.* sines and cosines) with a real temporal envelope that can be either exponentially decaying or expanding

Definition of the *Z*-transform

- **Recall that the DTFT is**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

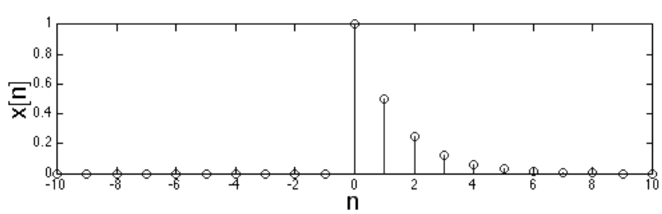
- **Since we are replacing (generalizing) the complex exponential building blocks $e^{j\omega n}$ by z^n, a reasonable extension of $X(e^{j\omega})$ would be**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- **Again, think of this as building up the time function by a weighted sums of functions z^n instead of $e^{j\omega n}$**

Computing the *Z-transform: an example*

- **Example 1:** Consider the time function $x[n] = \alpha^n u[n]$

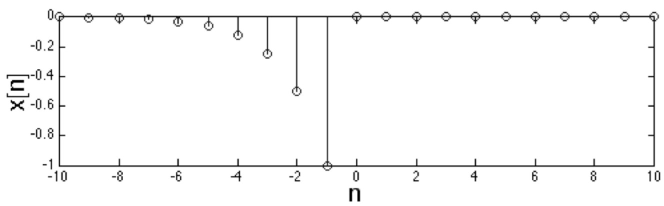


$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

Another example …

- **Example 2:** Now consider the time function $x[n] = -\alpha^n u[-n-1]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$$

- **Let** $l = -n; n = -\infty \Rightarrow l = \infty; n = -1 \Rightarrow l = 1$
- **Then,** $\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n = \sum_{l=1}^{\infty} -(z\alpha^{-1})^l = 1 - \sum_{l=0}^{\infty} (z\alpha^{-1})^l = 1 - \frac{1}{1 - z\alpha^{-1}} = \frac{1}{1 - \alpha z^{-1}}$

The importance of the region of convergence

- **Did you notice that the *Z*-transforms were identical for Examples 1 and 2 even though the time functions were different? Yes, indeed, very different time functions can have the same *Z*-transform! What's missing in this characterization? The region of convergence (ROC).**

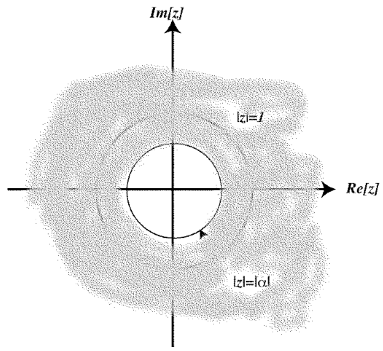
- **In Example 1, the sum** $X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$ **converges only for** $|z| > |\alpha|$

- **In Example 2, the sum** $X(z) = \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$ **converges only for** $|\alpha| > |z|$

- **So in general, we must specify not only the *Z*-transform corresponding to a time function, but its ROC as well.**

What shapes are ROCs for *Z*-transforms?

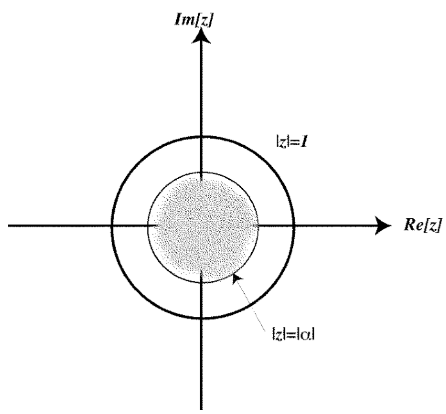
- **In Example 1, the ROC was $|z|>|\alpha|$ We can represent this graphically as:**



What shapes are ROCs for *Z*-transforms?

- **In Example 2, the ROC was $|z|<|\alpha|$ We can represent this graphically as:**

(ROC is shaded area)



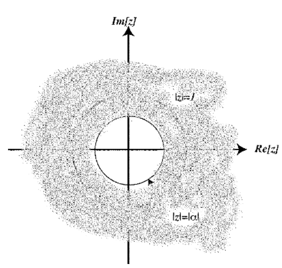
General form of ROCs

- **In general, there are four types of ROCs for *Z*-transforms, and they depend on the type of the corresponding time functions**
- **Four types of time functions:**
  - Right-sided
  - Left-sided
  - "Both"-sided (infinite duration)
  - Finite duration

Right-sided time functions

- **Right-sided time functions are of the form** $x[n] = 0, n < n_0$

(as in Example 1). ROCs are of the form $|\alpha| < |z|$

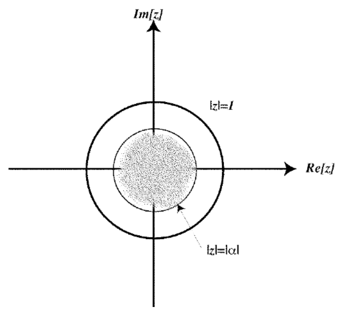


- **Comment:** All causal LSI systems have unit sample responses that are right-sided, although not all right-sided sample responses correspond to causal systems.

Left-sided time functions

- **Left-sided time functions are of the form** $x[n]=0, n>n_0$

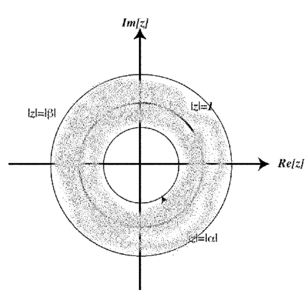
(as in Example 2). ROCs are of the form $|z|<|\alpha|$ **except that it is possible that they don't include** $z=0$



"Both"-sided (infinite-duration) time functions

- **Right-sided time functions are of the form** $x[n]\neq 0$ **for all** n

(as in $x[n]=\alpha^{|n|}, |\alpha|<1$ **). ROCs are of the form** $|\alpha|<|z|<|\beta|$**, an annulus bounded by** α **and** β**, exclusive.**



An example of a "both-sided" time function

- Consider the function $\alpha^n u[n] - \beta^n u[-n-1]$ with $|\alpha| < |\beta|$
- Using the results of Examples 1 and 2, we note that

$$X(z) = \frac{-1}{1-\beta z^{-1}} - \frac{1}{1-\alpha z^{-1}} = \frac{z(\alpha-\beta)}{(z-\alpha)(z-\alpha)}$$

- The ROC is $|\alpha| < z < |\beta|$, which is the region of "overlap" of the ROCs of the *z*-transforms of the two terms of the time function taken individually.

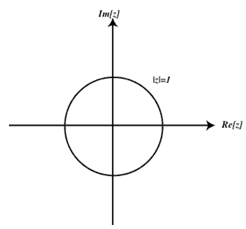
Finite-duration time functions

- Finite-duration time functions are of the form $x[n] \neq 0, \; n_1 < n < n_2$

ROCs include the entire *z*-plane except possibly $z = 0$

Stability and the ROC

- It can be shown that an LSI system is stable if the ROC includes the unit circle (UC), which is the locus of points for which $|z| = 1$



- Comment: this is exactly the same condition that is required for the DTFT $X(e^{j\omega})$ to exist

Causality, stability and the ROC

- Recall that for a system to be causal the sample response must be right-sided, and the ROC must be the outside of some circle.
- Hence, for a system to be both causal and stable, the ROC must be the outside of a circle that is inside the UC.
- In other words, if an LSI system is both causal and stable, the ROC will be of the form $|z| > \alpha$ with $|\alpha| < 1$

The inverse *Z*-transform

- **Did you notice that we didn't talk about inverse *z*-transforms yet?**
- **It can be shown (see the text) that the inverse *z*-transform can be formally expressed as**

$$x[n] = \frac{1}{2\pi j}\oint_c X(z) z^{n-1} dz$$

- **Comments:**
  - Unlike the DTFT, this integral is over a **complex variable,** z and we need complex residue calculus to evaluate it formally
  - The contour of integration, c, is a circle around the origin that lies inside the ROC
  - We will never need to actually evaluate this integral in this course … we'll discuss workaround techniques in the next class

Comparing the *Z*-transform with the LaPlace transform

Z-transforms:

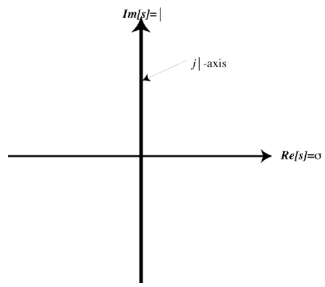
- **The *Z*-transform uses** $z^n = \rho e^{j\omega n} = \rho^n e^{j\omega n}$ **as the basic building block**
- **The DTFT exists if the ROC of the *Z*-transform includes the unit circle**
- **The DTFT equals the *Z*-transform evaluated along the unit circle,** $z = e^{j\omega}$
- **Causal and stable LSI systems have ROCs that are the outside of some circle that is to the inside of the unit circle**

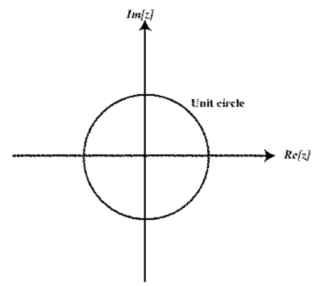
LaPlace transforms:

- **The LaPlace transform uses** $e^{st} = e^{(\sigma + j\Omega)t} = e^{\sigma t} e^{j\Omega t}$ **as the basic building block**
- **The CTFT exists of the ROC of the LaPlace transform includes the jΩ-axis,** $s = j\Omega$
- **The CTFT equals the LaPlace transform evaluated along the jΩ-axis,**
- **Causal and stable LTI systems have ROCs that are right-half planes bounded by a vertical line to the left of the jΩ-axis**

Mapping the *s*-plane to the *z*-plane

- **Map (*i.e.* warp conformally) the s-plane into the *z*-plane:**





- **Comments:**
  - jΩ-axis in s-plane maps to unit circle in z-plane
  - Right half of s-plane maps to outside of z-plane
  - Left half of z-plane maps to inside of s-plane

Summary - Intro to the *z*-transform

- **The *z*-transform is based on a generalization of the frequency representation used for the DTFT**
- **Different time functions may have the same *z*-transforms; the ROC is needed as well**
- **The ROC is bounded by one or more circles in the *z*-plane centered at its origin**
- **The shape of the ROC depends on whether the time function is right-sided, left-sided, infinite in duration, or finite duration**
- **An LSI system is stable if the ROC includes the unit circle**
- **The inverse *z*-transform can only be evaluated using complex contour integration**
- **The *z*-plane can be considered (in some ways) as a conformal mapping of the s-plane**