$\boxed{II}$ part (a) of problem 2.2

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{e}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{e}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Read pp 32-33 for more about Gram-Schmidt procedure.

$$\hat{\underline{x}}_1 = \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{\underline{x}}_2 = \underline{e}_2 - \langle \underline{e}_2, \underline{x}_1 \rangle \underline{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\hat{\underline{x}}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\hat{\underline{x}}_3 = \underline{e}_3 - \langle \underline{e}_3, \underline{x}_1 \rangle \underline{x}_1 - \langle \underline{e}_3, \underline{x}_2 \rangle \underline{x}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{6}} \right) \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{x}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\hat{\underline{x}}_4 = \underline{e}_4 - \langle \underline{e}_4, \underline{x}_1 \rangle \underline{x}_1 - \langle \underline{e}_4, \underline{x}_2 \rangle \underline{x}_2 - \langle \underline{e}_4, \underline{x}_3 \rangle \underline{x}_3$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \sqrt{2}\,\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{6}}\,\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore$$

$\underline{x}_4$ is unnecessary (This should have been expected since we already got 3 orthonormal vectors with 3 elements)

$$\text{basis} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$\boxed{2}$ Problem 2.8:

$$\underline{A} = \underline{I} - 2\,\underline{x}\,\underline{x}^T \qquad \text{where } \underline{x}^T\underline{x} = 1$$

ⓐ Prove $\underline{A}$ is symmetric ($\underline{A} = \underline{A}^T$)

$$\underline{A}^T = (\underline{I} - 2\underline{x}\,\underline{x}^T)^T = \underline{I}^T - (2\underline{x}\,\underline{x}^T)^T = \underline{I} - 2\underline{x}\,\underline{x}^T = \underline{A}$$

$\Rightarrow \underline{A}$ is symmetric

ⓑ prove $\underline{A}^2 = \underline{I}$

$$\underline{A}^2 = \underline{A}\,\underline{A} = (\underline{I} - 2\underline{x}\,\underline{x}^T)(\underline{I} - 2\underline{x}\,\underline{x}^T)$$

$$= \underline{I} - 2\underline{x}\,\underline{x}^T - 2\underline{x}\,\underline{x}^T + 4\underline{x}\,\underbrace{\underline{x}^T\underline{x}}_{1}\,\underline{x}^T$$

$$= \underline{I} - 4\underline{x}\,\underline{x}^T + 4\underline{x}\,\underline{x}^T = \underline{I}$$

$\therefore \underline{A}^2 = \underline{I}$

$\boxed{3.}$ Problem 2.17

$\underline{A}$ is a 3x3 matrix and has the following eigenvalues:

$\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 1$

If $\underline{A}\,\underline{e} = \lambda_A \underline{e}$, then $\underline{B}\,\underline{e} = (2\underline{A}^2 - 3\underline{I} + 6\underline{A}^{-1})\,\underline{e}$

$$= (2\lambda_A^2 - 3 + \frac{6}{\lambda_A})\,\underline{e}$$

$\therefore \lambda_B = 2\lambda_A^2 - 3 + \dfrac{6}{\lambda_A}$

$\therefore$ eigenvalues of $\underline{B}$ are $8, -7, 5$

$\therefore |\underline{B}| = \lambda_{B1}\lambda_{B2}\lambda_{B2} = -280$

[4] Problem 2.24

$$\underline{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$ we want to find $\underline{V}$

such that $\underline{V}^H \underline{A} \underline{V} = \underline{I}$

Since $\underline{A}$ is a symmetric matrix, its eigenvectors are orthogonal and can be normalized to have $\underline{E}^H \underline{E} = \underline{I}$. $\underline{\Lambda}$ can be split into two matrices whose diagonal entries are the square root of those in $\underline{\Lambda}$.

$$\underline{E}^H \underline{A} \underline{E} = \underline{\Lambda} = \underline{\Lambda}^{1/2} \underline{\Lambda}^{1/2}$$

$$\underline{\Lambda}^{-1/2} \underline{E}^H \underline{A} \underline{E} \underline{\Lambda}^{-1/2} = \underline{I}$$

$$(\underline{E} \underline{\Lambda}^{-1/2})^H \underline{A} (\underline{E} \underline{\Lambda}^{-1/2}) = \underline{I}$$

$$\therefore \underline{V} = \underline{E} \Lambda^{-1/2}$$

First, lets find $\underline{\Lambda}$

$$|\underline{A} - \lambda \underline{I}| = (2-\lambda)((2-\lambda)^2 - 1) - 0 + 1(0 - (2-\lambda))$$

$$= (2-\lambda)((2-\lambda)^2 - 2) = 0$$

$$\Rightarrow \lambda = 2, \ 2 \mp \sqrt{2}$$

$$\underline{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}$$

Eigenvectors can be found as follows:

$$(\underline{A} - \lambda \underline{I})\,\underline{e} = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = \underline{0} \implies \underline{e}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_1 = 2$$

$$\begin{bmatrix} -\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 1 \\ 1 & 1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = \underline{0} \implies \underline{e}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} \quad \lambda_2 = 2+\sqrt{2}$$

$$\begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & 1 \\ 1 & 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = \underline{0} \implies \underline{e}_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ \sqrt{2} \end{bmatrix} \quad \lambda_3 = 2-\sqrt{2}$$

$$\underline{V} = \underline{E}\,\underline{\Lambda}^{-1/2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2+\sqrt{2}}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2-\sqrt{2}}} \end{bmatrix}$$

$$\underline{V} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2\sqrt{2+\sqrt{2}}} & -\frac{1}{2\sqrt{2-\sqrt{2}}} \\ \frac{1}{2} & \frac{1}{2\sqrt{2+\sqrt{2}}} & -\frac{1}{2\sqrt{2-\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$$

5 (**problem 2.25**) Determine a transofrmation matrix V such that $V^H A V = I$ and $V^H B V = D$ where D is a diagonal matrix and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Here is the general procedure for solving this problem, as explained on pp. 50-51 of the handout:

1) find the diagonal eigenvalue matrix Λ_A and eigenvector matrix E_A for matrix A.

2) find an intermediate matrix $C = H^H B H$, where $H = E_A \Lambda_A^{-1/2}$ as in problem 4.

3) find the eigenvectors E_c of matrix C.

4) find $V = HE_C$.

Don't even think of doing all this algebra by hand. Here is a Matlab program.

```
A = [1 0 0; 0 2 -1; 0 -1 3];
B = [2 1 0; 1 2 0; 0 0 4];
[E,L]=eig(A);
Ls = sqrtm(inv(L));
H=E*Ls;
C=H'*B*H;
[Ec,v]=eig(C);
V=H*Ec;
```

Here are the results:

$$\boxed{V = \begin{bmatrix} 0.7337 & -0.5543 & 0.3930 \\ 0.4797 & 0.2381 & -0.5597 \\ 0.3213 & 0.5259 & 0.1418 \end{bmatrix}}$$

To check V, we can make sure that $V^H A V = I$ and $V^H B V = D$.

$$V'AV = \begin{bmatrix} 1.0000 & 0.0000 & -0.0000 \\ 0.0000 & 1.0000 & -0.0000 \\ -0.0000 & -0.0000 & 1.0000 \end{bmatrix} \qquad V'BV = \begin{bmatrix} 2.6537 & -0.0000 & 0.0000 \\ -0.0000 & 1.5703 & -0.0000 \\ 0.0000 & -0.0000 & 0.5759 \end{bmatrix}$$