Optimal Memoryless Relays with Noncoherent Modulation

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***Abstract*—We derive optimal memoryless relays using noncoherent modulation over additive white Gaussian noise (AWGN) channels with or without fading. The derivation is flexible, as it can be applied to any binary hypothesis test regarding the observations at the relay. We investigate several channels, including random phase and fading, and apply different modulation schemes, namely on-off-keying (OOK) and orthogonal frequency-shift-keying (FSK). We find that at low signal-to-noise ratio (SNR) the relay censors its observation, as it only transmits at non-zero energy if the observations seem reliable. Compared to the known results that optimal memoryless relays using coherent BPSK are combinations of soft-information and hard-limiter [1]–[3], the noncoherent relays have considerably less emphasis on soft-information and converge much faster to the hard-limiter.**

I. Introduction

We study a simple scenario, where a relay or sensor node processes its observations to a sufficient statistic y, and then forwards $U(y)$ to a receiver or fusion center, where $U(y)$ is a non-linear function in general. We want to find the optimal function $U(y)$ that minimizes the probability of error at the receiver, subject to an average transmission power constraint at the relay. We find that the optimal function naturally follows the chosen modulation on a single link, but with judicious power usage depending on the quality of the measurements. For example, if the carrier phase is not estimated at the receiver, the optimal $U(y)$ naturally follows as on-off-keying (OOK), or if the instant channel gain is not known at the relay, the fading statistics are accounted for in the optimal $U(y)$ as well. An overview of possible combinations of modulation and channel fading is given in Table I, where the solutions will be derived in the main body of this paper. Note that each solution will be characterized by a nonlinear function that is applied to process the data.

Our contributions in this paper are:

- We derive optimal memoryless relay functions $U(y)$, which minimize the probability of error at the receiver for different settings, including OOK and FSK in both fading and nonfading channels.
- By applying the relay function $U(y)$ to a sufficient statistic of the observations, y, our solution can be easily generalized to arbitrary observations, e.g., mean-shift with arbitrary noise, variance change, multiple observations.
- We find that by minimizing the probability of error at the receiver, the optimal $U(y)$ automatically accounts for unknown phase and/or channel statistics, which has not been considered in previous treatments of this topic.
- We find that there is a simple link between the SNR optimal and the error-performance optimal solutions, where the latter consists of an additional nonlinear processing to the former before transmission. Therefore a suitable description of the probability of error optimal scheme is: Estimate-*Process*-and-Forward (EPnF).

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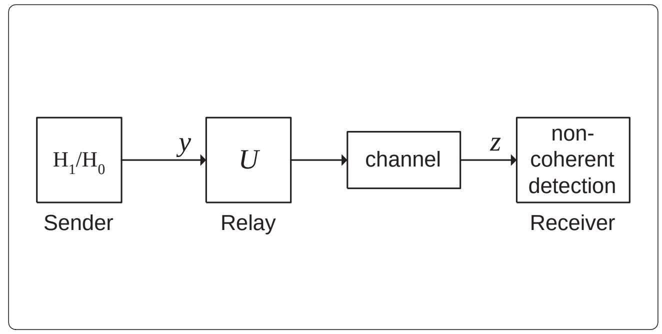

Fig. 1. The relay framework; the observation y at the relay includes some approriate processing and the channel is AWGN with or without fading.

In our numerical examples we observe that compared to coherent modulation, the optimal forwarding function for noncoherent modulation has some distinct differences. While for coherent modulation on an AWGN channel the optimal forwarding function is the hyperbolic tangent (SNR optimal) [2], [3], which clearly combines soft-information as in Amplify-and-Forward (AnF) with a hard-limiter as in Decode-and-Forward (DnF), the optimal function using noncoherent modulation is different. Instead of soft-information for low SNR, it shows that using noncoherent modulation it is preferable to conserve energy and transmit nothing in case of unreliable observations. Therefore there is considerable less emphasis on soft-information and the forwarding function converges much faster to a hard-limiter type like DnF. This is also reflected in performance evaluation, as the optimal relay function has little gain over the DnF. Other approaches, like the SNR optimal Estimate-and-Forward (EnF), do not account for channel statistics on the forward channel and show degraded performance in most scenarios, as the decision statistic at the output of an energy detector has no apparent resemblance to a Gaussian distribution.

The rest of this paper is organized as follows, in Section II

	BPSK		OOK		FSK	
Channel type	non-fading	fading	fading	non-fading	fading	non-fading
Non-linearity	$\mathcal{W}(\cdot)$	$\mathcal{F}_1(\cdot)$	$[\mathcal{W}(\cdot)]^{-1}$	$\mathcal{V}(\cdot)$	$\sqrt{(\cdot)}$	$\log(\cdot)$
Solution presented in	[1, Eq.(3)]	[4, Eq.(A.64)]	(12)	(20)	(30)/(31)	(33)/(34)

TABLE I

THE SOLUTIONS FOR THE OPTIMAL $U(y)$ DEPEND ON THE MODULATION AND CHANNEL; EACH CASE IS CHARACTERIZED BY A PARTICULAR NON-LINEARITY THAT IS APPLIED TO THE MMSE ESTIMATE.

we derive the optimal forwarding functions using on-off-keying (OOK) modulation. Next we derive optimal forwarding functions for orthogonal frequency-shift-keying (FSK) in Section III. We analyze the results and compare it to other known approaches in Section IV. Then we numerically evaluate the performance in Section V. Last we conclude in Section VI.

II. Error-Optimal Forwarding using On-Off-Keying

A. Preliminaries and Variational Approach

A relay observes a binary hypothesis H_0/H_1, processes its observations to a sufficient statistic y, then forwards y over a (possibly fading) additive Gaussian noise (AWGN) channel to a receiver, subject to an average-power constraint; Figure 1 shows the generic relay framework. Based upon the received signal z, the receiver must make a decision regarding H_0/H_1. We are interested in finding the memoryless, generally non-linear forwarding function $U(y)$, which minimizes the probability of error at the receiver. The argument y can be any sufficient statistic of the observations made at the relay, but for simpler presentation we choose the likelihood ratio (LR) or a monotonic function thereof.

The conditional probability density functions (PDFs) of y given H_0 and H_1, are abbreviated as $f_0(y)$ and $f_1(y)$ respectively. When choosing y as the LR or a monotonic transformation, we can assume w.l.o.g. $f_1(y) > f_0(y)$ for $y > \tau$. For ease of presentation we will adopt several assumptions:

Assumption 1 *Due to the unknown carrier phase, the optimal decision statistic at the receiver will be a function of the received energy.*

Assumption 2 *We assume that $U(y)$ is a monotonic function in y.*

Assumption 3 *The relay must obey the following average-power constraint:*

$$\mathrm{E}\left[|U(y)|^2\right] \leq P. \tag{1}$$

Comments: Assumption 1 is justified, since if the carrier phase is not estimated, the uncompensated phase of the signal will be uniformly distributed; therefore the amplitude or power are sufficient statistics. For Assumption 2 we can argue that we would not expend more energy, if we were less sure. Finally the energy-constraint in Assumption 3 is in the average sense, as sensor networks usually mean to conserve battery life.

Combining Assumption 1 and Assumption 2, the optimal decision at the receiver will be a threshold test on the received signal power:

$$d(z) = \begin{cases} H_1 & |z|^2 > \Gamma \\ H_0 & \text{otherwise} \end{cases} \tag{2}$$

With this the probability of a decision error at the receiver can be expressed as

$$\Pr(e) = \frac{1}{2}\left(\Pr\left\{|z|^2 > \Gamma \middle| H_0\right\} + \Pr\left\{|z|^2 < \Gamma \middle| H_1\right\}\right) \tag{3a}$$

$$= \frac{1}{2}\left(\Pr\left\{|z|^2 > \Gamma \middle| H_0\right\} + 1 - \Pr\left\{|z|^2 > \Gamma \middle| H_1\right\}\right), \tag{3b}$$

where we assume equally likely hypotheses H_0/H_1. We want to find the optimal relay function $U(y)$, in the sense of minimizing $\Pr(e)$ subject to the average power constraint in Assumption 3.

To find the optimal relay function, we have to rewrite $\Pr(e)$ as a function of $U(y)$. We achieve this by rewriting the probability of exceeding the threshold at the receiver, conditioned on the true hypothesis

$$\Pr\left(|z|^2 > \Gamma \middle| H_i\right) = \int_{\mathbb{Y}} \Pr\left\{|z|^2 > \Gamma \middle| U(y)\right\} f_i(y)\, dy, \tag{4}$$

where we denote the support of y as $\mathbb{Y}$. Inserting this into (3b) and defining the difference $\Delta_f(y) := f_1(y) - f_0(y)$ to abbreviate notation,

$$\Pr(e) = \frac{1}{2}\left(1 - \int_{\mathbb{Y}} \Pr\left\{|z|^2 > \Gamma \middle| U(y)\right\} \Delta_f(y)\, dy\right). \tag{5}$$

Using this expression, we can define a Lagrangian function $\mathcal{L}$ to incorporate the average power constraint in (1),

$$\mathfrak{L} = \Pr(e) + \lambda \int_{\mathbb{Y}} \left(|\mathbf{U}(y)|^2 - P\right) \Sigma_f(y)\, dy. \tag{6}$$

where we now define the sum $\Sigma_f(y) := f_0(y) + f_1(y)$. We combine the integrals to

$$I(U,\lambda) = -\frac{1}{2}\Pr\left\{|z|^2 > \Gamma \middle| U(y)\right\} \Delta_f(y) + \lambda\left(|\mathbf{U}(y)|^2 - P\right)\Sigma_f(y), \tag{7}$$

which can be minimized using a variational approach as defined by calculus of variations, see e.g. [5]. This amounts to taking the derivative of I with respect to U to find a stationary point as necessary condition,

$$\frac{\partial I(U,\lambda)}{\partial U} = 0, \tag{8}$$

then use (1) to solve for λ and check if the stationary point is in fact a minimum using a sufficient condition.

B. Fading AWGN Channel

We define the Rayleigh fading AWGN channel as

$$z = hU(y) + w \tag{9}$$

where h is a complex Gaussian fading coefficient of unit norm and w is complex AWGN of power N_0. To express the probability to exceed the threshold at the receiver as in (4), we see that $|z|^2$ given y is distributed exponentially with mean $|U(y)|^2 + N_0$. Therefore, the probability of exceeding the threshold, conditioned on a particular y, is

$$\Pr\left\{|z|^2 > \Gamma \middle| U(y)\right\} = \exp\left(-\frac{\Gamma}{|U(y)|^2 + N_0}\right). \tag{10}$$

Inserting this into (7) and taking the derivative, we find

$$\frac{\partial I}{\partial |U|} = 2|U(y)|\Bigg[\lambda\Sigma_f(y) - \frac{\Gamma\Delta_f(y)}{\left(|U(y)|^2 + N_0\right)^2}\exp\left(-\frac{\Gamma}{|U(y)|^2 + N_0}\right)\Bigg], \tag{11}$$

where we took the derivative with respect to $|U|$, since (10) does not depend on the phase of U. We immediately find one stationary point as $|U(y)| = 0$. After simplification, we find another stationary point as:

$$|U(y)|^2 = -\frac{\Gamma}{2}\left[\mathcal{W}\left(-\frac{1}{2}\sqrt{\lambda\Gamma\cdot\frac{\Sigma_f(y)}{\Delta_f(y)}}\right)\right]^{-1} - N_0 \tag{12}$$

which is only defined for $\frac{\Delta_f(y)}{\Sigma_f(y)} \geq \frac{\lambda N_0^2}{\Gamma} e^{\frac{\Gamma}{N_0}}$, and $\mathcal{W}(\cdot)$ refers to the Lambert-$\mathcal{W}$ function; w.l.o.g. we define $U(y)$ as the positive, real square root and assume $\lambda > 0$ — for $\lambda < 0$ the hypotheses are swapped. We refer the reader to [4] for details of the justifications when which solution is optimal using a sufficient condition and how the unknown decision threshold Γ and power constraint λ can be efficiently determined.

C. Non-fading AWGN Channel with Random Carrier Phase

The non-fading AWGN channel is defined as,

$$z = e^{j\phi}U(y) + w, \tag{13}$$

with $\phi \in [0,\, 2\pi]$ the random carrier phase and w complex Gaussian noise of power N_0. Accordingly, $|z|^2$ conditioned on a certain y will be non-central chi-square distributed with non-centrality parameter $|U(y)|^2$ and the phase of z uniform between zero and 2π. Therefore we can write the probability of $|z|^2$ to exceed the threshold conditioned on a certain value of $U(y)$ as the complementary probability distribution function of a non-central chi-squared random variable,

$$\begin{aligned}\Pr\left\{|z|^2 > \Gamma \middle| U(y)\right\} &= \int_{\Gamma}^{\infty} \frac{1}{N_0}\exp\left(-\frac{|U(y)|^2 + \zeta}{N_0}\right) I_0\left(\frac{\sqrt{|U(y)|^2\zeta}}{N_0/2}\right) d\zeta \\ &= \mathcal{Q}\left(\sqrt{\frac{|U(y)|^2}{N_0/2}}, \sqrt{\frac{\Gamma}{N_0/2}}\right),\end{aligned} \tag{14}$$

where Marcum's $\mathcal{Q}$-function is defined as [6]:

$$\mathcal{Q}\left(a, b\right) = \int_b^{\infty} \exp\left(-\frac{a^2 + x^2}{2}\right) x I_0(ax)\, dx. \tag{15}$$

Inserting this into (7) and taking the partial derivative with respect to $|U|$:

$$\begin{aligned}\frac{\partial I}{\partial |U|} = &-\frac{\sqrt{\Gamma}}{N_0}\Delta_f(y) I_1\left(\frac{\sqrt{|U(y)|^2\Gamma}}{N_0/2}\right) \\ &\times \exp\left(-\frac{|U(y)|^2 + \Gamma}{N_0}\right) + \lambda 2|U(y)|\Sigma_f(y),\end{aligned} \tag{16}$$

where we use the following result of [7]:

$$\frac{\partial \mathcal{Q}(a, b)}{\partial a} = bI_1(ab)\exp\left(-\frac{a^2 + b^2}{2}\right). \tag{17}$$

Therefore we find one stationary point again as $|U(y)| = 0$, since $I_1(0) = 0$, and another as the solution to the following equation:

$$\frac{N_0/2}{\sqrt{|U(y)|^2\Gamma}} I_1\left(\frac{\sqrt{|U(y)|^2\Gamma}}{N_0/2}\right)\exp\left(-\frac{|U(y)|^2 + \Gamma}{N_0}\right) = \frac{\lambda N_0^2}{\Gamma}\frac{\Sigma_f(y)}{\Delta_f(y)}. \tag{18}$$

We define the function $\mathcal{V}(z, b)$ implicitly as

$$z = \frac{1}{b\mathcal{V}(z, b)} I_1\left[b\mathcal{V}(z, b)\right]\exp\left(-\frac{\mathcal{V}(z, b)^2 + b^2}{2}\right), \tag{19}$$

and with that the solution to (18) can be expressed as:

$$|U(y)|^2 = \frac{N_0}{2}\left[\mathcal{V}\left(\frac{\lambda N_0^2}{\Gamma}\frac{\Sigma_f(y)}{\Delta_f(y)}, \sqrt{\frac{2\Gamma}{N_0}}\right)\right]^2 \tag{20}$$

which only exists for $\frac{\Delta_f(y)}{\Sigma_f(y)} \geq \frac{2\lambda N_0^2}{\Gamma} e^{\frac{\Gamma}{N_0}}$, since the left-hand-side of (18) takes its maximum for $|U(y)| = 0$.

III. Error-Optimal Forwarding using Orthogonal Frequency-Shift-Keying

A. Modifications using Two Orthogonal Basis Functions

The system setup stays largely unchanged as in Fig. 1. Assuming two orthogonal basis functions available, we extend the previous notation to the vector case: $U(y) \rightarrow \mathbf{U}(y)$, $w \rightarrow \mathbf{w}$ and $z \rightarrow \mathbf{z}$ become two-dimensional vectors, e.g.,

$$\mathbf{U}(y) = [U_0(y),\ U_1(y)]^T \tag{21}$$

and the same notation for the other vectors.

Since there are two basis functions, Assumption 1 implies the optimal decision statistic is a comparison of the received power on the different basis functions, c.f. [6]:

$$d(\mathbf{z}) = \begin{cases} H_0 & |z_0|^2 > |z_1|^2 \\ H_1 & \text{otherwise} \end{cases} \tag{22}$$

With this, the probability of a decision error at the receiver is slightly different from (3b)

$$\Pr(e) = \frac{1}{2}\left(\Pr\left\{|z_0|^2 < |z_1|^2 \middle| H_0\right\} + \Pr\left\{|z_0|^2 > |z_1|^2 \middle| H_1\right\}\right)$$
$$= \frac{1}{2}\left(1 - \Pr\left\{|z_0|^2 > |z_1|^2 \middle| H_0\right\} + \Pr\left\{|z_0|^2 > |z_1|^2 \middle| H_1\right\}\right).$$

To express $\Pr(e)$ as function of $\mathbf{U}(y)$ we use

$$\Pr\left\{|z_0|^2 > |z_1|^2 \middle| H_i\right\} = \int_{\mathbb{Y}} \Pr\left\{|z_0|^2 > |z_1|^2 \middle| \mathbf{U}(y)\right\} f_i(y)\, dy \tag{24}$$

which takes the place of (4). Otherwise the variational approach follows completely the same pattern as in Section II. Defining a Lagrangian function, combining the integrals, we arrive at the following gradient

$$\nabla I(U,\lambda) = \begin{bmatrix} \frac{\partial I(U,\lambda)}{\partial U_0} & \frac{\partial I(U,\lambda)}{\partial U_1} \end{bmatrix}^T = \mathbf{0} \tag{25}$$

to find a stationary point.

B. Fading AWGN Channel

The channel model for this scenario is an extension of that in Section II-B,

$$\mathbf{z} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \mathbf{U}(y) + \mathbf{w}, \tag{26}$$

where h_1 and h_2 are independent Gaussian variances of zero mean and unit variance, and $\mathbf{w}$ is additive white Gaussian noise with covariance $N_0\mathbf{I}$.

From Equation (26), it can be seen that z_i given y is distributed Gaussian with zero mean and variance $|U_i(y)|^2 + N_0$. We calculate:

$$\Pr\left\{|z_0|^2 > |z_1|^2 \middle| \mathbf{U}(y)\right\} = \frac{|U_0|^2 + N_0}{|U_0|^2 + |U_1|^2 + 2N_0}, \tag{27}$$

where we temporarily dropped the arguments of the U_i's for more compact notation. We calculate the partial derivatives of the gradient:

$$\frac{\partial I}{\partial |U_0|} = 2|U_0| \left(\lambda \Sigma_f(y) + \Delta_f(y) \frac{|U_1|^2 + N_0}{\left(|U_0|^2 + |U_1|^2 + 2N_0\right)^2}\right)$$
$$\frac{\partial I}{\partial |U_1|} = 2|U_1| \left(\lambda \Sigma_f(y) - \Delta_f(y) \frac{|U_0|^2 + N_0}{\left(|U_0|^2 + |U_1|^2 + 2N_0\right)^2}\right)$$

Both partial derivatives have a common root $|U_i| = 0$ and a second root determined by the following equations:

$$\frac{|U_1|^2 + N_0}{\left(|U_0|^2 + |U_1|^2 + 2N_0\right)^2} = -\lambda \frac{\Sigma_f(y)}{\Delta_f(y)} \tag{28}$$
$$\frac{|U_0|^2 + N_0}{\left(|U_0|^2 + |U_1|^2 + 2N_0\right)^2} = \lambda \frac{\Sigma_f(y)}{\Delta_f(y)} \tag{29}$$

We observe that the left-hand-side of (28)/(29) is always positive, while the sign of the right-hand side depends on the particular y. This leads to the conclusion that the equations do not have a common solution. Therefore there are only three posibble stationary points:

a) $|U_0| = |U_1| = 0$: Both outputs are zero.

b) $|U_1| = 0$: Solving (28), we obtain $|U_0|$ as:

$$|U_0(y)|^2 = N_0 \left(\sqrt{\max\left\{-\frac{1}{\lambda N_0}\frac{\Delta_f(y)}{\Sigma_f(y)}, 4\right\}} - 2\right) \tag{30}$$

c) $|U_0| = 0$: Solving (29), we obtain $|U_1|$ as:

$$|U_1(y)|^2 = N_0 \left(\sqrt{\max\left\{\frac{1}{\lambda N_0}\frac{\Delta_f(y)}{\Sigma_f(y)}, 4\right\}} - 2\right) \tag{31}$$

C. Non-fading AWGN Channel with Random Carrier Phase

The constant AWGN channel for orthogonal FSK is a straightforward extension of Section II-C to the vector case,

$$\mathbf{z} = \begin{pmatrix} e^{j\phi_1} & 0 \\ 0 & e^{j\phi_2} \end{pmatrix} \mathbf{U}(y) + \mathbf{w} \tag{32}$$

where $\phi_i \in [0, 2\pi]$ is the random carrier phase and $\mathbf{w}$ additive white Gaussian noise with covariance $N_0\mathbf{I}$.

To render this problem mathematically tractable, we adopt the following assumption motivated by the result of the previous section:

Assumption 4 *The optimal relay function $\mathbf{U}(y)$, mapping to two orthogonal basis functions, has only one non-zero output at any one time.*

Given Assumption 4, the decision statistic at the receiver compares a Rician to a Rayleigh distributed random variable, depending on which component of $\mathbf{U}(y)$ has non-zero energy. This coincides with the probability of error for binary orthogonal FSK, which can be found in standard textbooks [6],

$$\Pr\left\{|z_0|^2 > |z_1|^2 \middle| \mathbf{U}\right\} = \begin{cases} 1 - \frac{1}{2}\exp\left(-\frac{|U_0|^2}{2N_0}\right), & |U_0|^2 > 0 \\ \frac{1}{2}\exp\left(-\frac{|U_1|^2}{2N_0}\right), & |U_1|^2 > 0 \end{cases}$$

Taking the partial derivatives of I using the above definition, we find:

$$\frac{\partial I}{\partial |U_0|} = 2|U_0(y)| \left[\lambda \Sigma_f(y) + \frac{\Delta_f(y)}{8N_0} \exp\left(-\frac{|U_0(y)|^2}{2N_0}\right)\right]$$
$$\frac{\partial I}{\partial |U_1|} = 2|U_1(y)| \left[\lambda \Sigma_f(y) - \frac{\Delta_f(y)}{8N_0} \exp\left(-\frac{|U_1(y)|^2}{2N_0}\right)\right]$$

Again there are three posibble stationary points:

a) $|U_0| = |U_1| = 0$: Both outputs are zero.

b) $|U_1| = 0$: We obtain the non-zero $|U_0|$ as:

$$|U_0(y)|^2 = 2N_0 \log\left(\max\left\{-\frac{1}{8N_0\lambda}\frac{\Delta_f(y)}{\Sigma_f(y)}, 1\right\}\right) \tag{33}$$

c) $|U_0| = 0$: We obtain the non-zero $|U_1|$ as:

$$|U_1(y)|^2 = 2N_0 \log\left(\max\left\{\frac{1}{8N_0\lambda}\frac{\Delta_f(y)}{\Sigma_f(y)}, 1\right\}\right) \tag{34}$$

IV. Analysis and Comparison of Results

A. Comparison with Other Relays

Other relays are usually specified in terms of the "signal in noise" scenario; to come to a common notation we define a binary signal $\mathbf{s}$ according to the employed modulation, e.g., for FSK the $\mathbf{s}_i$, $i = 0, 1$, are the basis vectors of the two channels (see also Section V-B), for BPSK and OOK s reduces to a scalar $\{\pm 1\}$ or $\{0, 1\}$ respectively. This signal $\mathbf{s}$ is observed at the relay as

$$\mathbf{x} = \mathbf{s} + \mathbf{v}, \tag{35}$$

where $\mathbf{v}$ is complex noise of power $2\sigma_v^2$ and y is accordingly a function of $\mathbf{x}$. For ease of representation we also define:

$$\gamma_{\text{obs}} = \frac{\mathrm{E}\left[|\mathbf{s}|^2\right]}{2\sigma_v^2}, \quad \gamma_{\text{fwd}} = \frac{P}{N_0}, \tag{36}$$

which are the SNR's on the sender-relay and relay-receiver channel respectively.

Due to our variational approach, which directly minimizes the probability of error, our work can be seen as an extension of [1] to the case of noncoherent modulation. Other known approaches for the coherent relay are:

- *Decode-and-Forward* (DnF) The relay decides on the hypothesis that minimizes the probability of error at the relay, and forwards this decision with constant power:
$$\mathbf{U}_{\text{DnF}}(\mathbf{x}) = \lambda \hat{\mathbf{s}} = \lambda \arg\max_{\mathbf{s}} f(\mathbf{x} \mid \mathbf{s})$$
- *Amplify-and-Forward* (AnF) The relay amplifies the received values by a constant factor:
$$\mathbf{U}_{\text{AnF}}(\mathbf{x}) = \lambda \mathbf{x}$$
- *Estimate-and-Forward* (EnF) As described in [3], forwarding the minimum mean-square error (MMSE) estimate subject to the power constraint amounts to a linearly scaled version of the conditional expectation:
$$\mathbf{U}_{\text{EnF}}(\mathbf{x}) = \lambda\, \mathrm{E}\left[\mathbf{s} \mid \mathbf{x}\right]$$

As a comparison we plot realizations of all four forwarding functions on the noncoherent fading channel for OOK modulation, see Fig 2. Contrary to optimal forwarding functions using coherent modulation, c.f., [1]–[3], $\mathbf{U}(y)$ is not a combination of soft-information (AnF) and delimiter (DnF), but a delimiter with a "cut-off" and a fairly short transition.

The optimal function is most similar to DnF, the only other function that displays a similar "cut-off". However, the "cut-off" point of DnF is lower, expending energy more often. This can be seen as a result of local decisions of DnF minimizing the decision error *at the relay*, without regard for optimal energy usage. Instead the optimal function has a higher threshold to expend energy, but can then allocate more power when the observations seem significant.

Interestingly EnF shares the same transition with the optimal function, but has no "cut-off". Although for higher SNR, γ_{fwd}, both EnF and the optimal function quickly converge to the DnF delimiter function, for low SNR EnF shows an energy inefficient behavior.

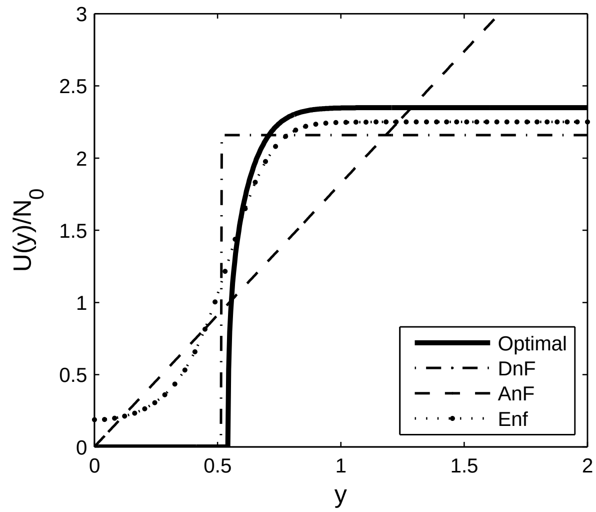

Fig. 2. Plot of the OOK noncoherent forwarding function for the Rayleigh fading AWGN channel as functions of a sufficient statistic y; $\gamma_{\text{obs}} = 10$ dB and $\gamma_{\text{fwd}} = 3$ dB.

B. Estimate-Process-and-Forward

In the case of coherent modulation, where $s_i = \pm 1$, the SNR optimal approach EnF reduces to

$$\mathrm{E}\left[\mathbf{s} \mid \mathbf{x}\right] = \frac{f_1(y) - f_0(y)}{f_1(y) + f_0(y)} = \frac{\Delta_f(y)}{\Sigma_f(y)}. \tag{37}$$

Interestingly, not only the optimal functions derived in this work use (37) as the input to a following non-linear function, but a similar observation can be made in [1] for the coherent case. Changing [1, Eq. (3)] to our notation, we have

$$|U(y)|^2 = \frac{N_0}{2}\mathcal{W}\left(\frac{1}{2\pi}\left(\frac{1}{\lambda N_0}\frac{\Delta_f(y)}{\Sigma_f(y)}\right)^2\right), \tag{38}$$

which uses the MMSE estimate $\mathrm{E}\left[\mathbf{s} \mid \mathbf{x}\right]$ as its input.

With these observations we determine the difference between the SNR optimal and probability-of-error optimal relays as an additional processing step using a non-linear function, optimally mapping the MMSE estimate to the output modulation. Using a similar expression we call the optimal relay function Estimate-*Process*-and-Forward (EPnF).

V. Numerical Results

Due to space limitations, we will only consider the performance of two cases, namely OOK over a fading channel (as plotted in Fig. 2) and FSK over a non-fading channel. For more detailed results we refer to [4].

A. On-Off Keying

The sender-relay channel model is defined analogously to (9) as

$$x = h_r s + v, \quad s \in \{0, 1\} \tag{39}$$

where h_r is the unit variance, complex Gaussian channel coefficient and v AWGN of power $2\sigma_v^2$. As sufficient statistic we choose the power of x, $y = |x|^2$. Accordingly y is

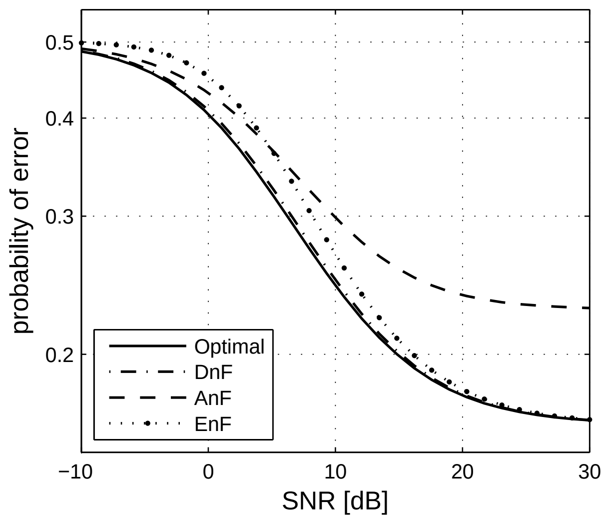

Fig. 3. Performance of different forwarding functions for OOK modulation over the Rayleigh fading AWGN channel; $\gamma_{\text{fwd}} = 6$ dB.

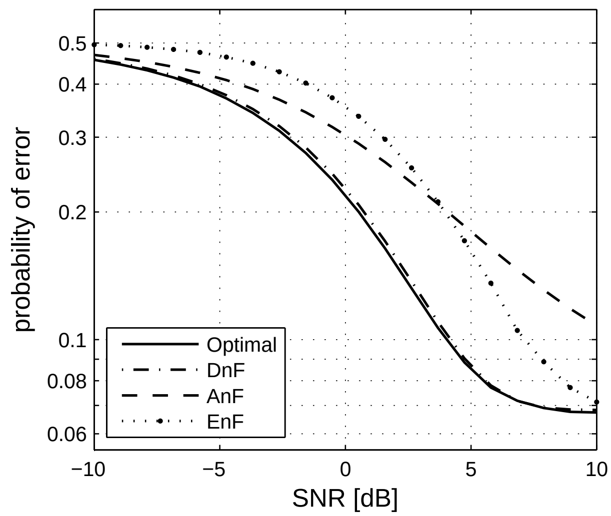

Fig. 4. Performance of different forwarding functions over a constant AWGN channel with random carrier phase using orthogonal FSK; for $\gamma_{\text{fwd}} = 6$ dB.

distributed exponentially with mean $2\sigma_v^2$ or $1 + 2\sigma_v^2$ under hypotheses H_0/H_1 respectively.

We compare the performance of the different forwarding functions. We keep the SNR on the relay-receiver link constant at $\gamma_{\text{fwd}} = 6$ dB and vary γ_{obs}. The probability of error is plotted in Fig. 3; surprisingly DnF outperforms EnF and is generally very close to the optimal performance. In fact EnF is even worse than AnF for small SNR; going back to Fig. 2, we see that EnF always expends energy, since there is always a non-zero probability that H_1 was observed at the relay. When reducing γ_{fwd} to 3 dB, the difference between DnF and the optimal performance is larger, but the performance is very limited due to the generally low performance of noncoherent modulation.

B. Frequency Shift Keying

We start with the results from Section III-C, the channel model is analogous to (32)

$$\mathbf{x} = \begin{pmatrix} e^{j\theta_1} & \\ & e^{j\theta_2} \end{pmatrix} \mathbf{s} + \mathbf{v}, \quad \mathbf{s} \in \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \tag{40}$$

where θ_i is the random phase and $\mathbf{v}$ is the complex AWGN as before. Accordingly the elements of $\mathbf{x} = [x_0, x_1]^T$ are distributed Rayleigh and Rician respectively and vice versa depending on $\mathbf{s}$. As sufficient statistic y we choose the log-likelihood ratio, which can be simplified to the following [6],

$$y = \log \left[\frac{f_1(x_0, x_1)}{f_0(x_0, x_1)} \right] = \log \left[\frac{I_0 \left(|x_1| / \sigma_v^2 \right)}{I_0 \left(|x_0| / \sigma_v^2 \right)} \right]. \tag{41}$$

Unfortunately the PDF of this random variable is non-trivial to derive, and we have to evaluate the performance using a two dimensional integral over (x_0, x_1).

We compare the performance of the different forwarding functions, keeping the SNR on the relay-receiver link constant at $\gamma_{\text{fwd}} = 6$ dB and varying γ_{obs}. The error probabilities are plotted in Fig. 4; the trends are as before, but we notice that compared to Fig. 3 the differences in performance are much larger for EnF and AnF, which shows a strongly inefficient behavior.

VI. CONCLUSION

We investigated forwarding using different types of noncoherent modulation. We derived optimal memoryless forwarding functions and compared them to known forwarding functions, namely Decode-and-Forward, Amplify-and-Forward and Estimate-and-Forward. We found that Decode-and-Forward performs very close to the optimal forwarding function for any reasonable SNR, while both outperform Amplify-and-Forward and Estimate-and-Forward. This fact that EnF does not perform close to optimal can be explained by linking the probability of error optimal formulation to EnF: there exists an additional non-linear processing on top of the MMSE estimate. While this additional processing step seems to be minor in the coherent BPSK case, for noncoherent modulation it is necessary to achieve efficient transmission.

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