Algebraic Signal Processing Theory: 1-D Nearest-Neighbor Models

Aliaksei Sandryhaila, *Member, IEEE*, Jelena Kovačević, *Fellow, IEEE*, and Markus Püschel, *Senior Member, IEEE*

***Abstract*—We present a signal processing framework for the analysis of discrete signals represented as linear combinations of orthogonal polynomials. We demonstrate that this representation implicitly changes the associated shift operation from the standard time shift to the nearest-neighbor shift introduced in this paper. Using the algebraic signal processing theory, we construct signal models based on this shift and derive their corresponding signal processing concepts, including the proper notions of signal and filter spaces, z-transform, convolution, spectrum, and Fourier transform. The presented results extend the algebraic signal processing theory and provide a general theoretical framework for signal analysis using orthogonal polynomials.**

***Index Terms*—Signal model, Fourier transform, orthogonal polynomials, Hermite polynomials, Legendre polynomials, Laguerre polynomials, signal representation, filter, shift, convolution, algebra, module.**

I. Introduction

Traditional discrete-time signal processing (SP) is based on a set of fundamental concepts including time shift, signals, filters, z-transform, convolution, spectrum, and Fourier transform. These concepts come in two variants: one for infinite signals and one for finite (usually periodically extended) signals. The algebraic signal processing (ASP) theory [1] has shown that the exact form of these and other concepts can be derived from the discrete-time shift operator, and that the same derivation can be used to obtain SP frameworks for shift operators different from the time shift.

As an illustration, consider Table I. The first column of the table lists several basic SP concepts. The second column shows their instantiation for discrete-time SP. The first concept is the shift, visualized as an operator moving a time point to the next one. For simplicity, we denote the shift with x instead of z^{-1} as usually done in time SP [2]. Solving the corresponding equation $xp_k = p_{k+1}$ yields the basis polynomials $p_k = x^k$ and leads to the traditional z-transform (by substituting $x = z^{-1}$) and linear convolution. The Fourier transform can then be constructed as a projection of a signal on the eigenfunctions of the shift operator. This brief discussion omits many details that can be found in [1].

One interesting aspect of this approach is that one can think of changing the notion of the shift operation to obtain SP theories that are different from the standard time SP, but still possess all relevant concepts. This was done in [3], [4], where a theory of 1-D space[1] SP was derived based on the symmetric space shift operator shown in the third column of Table I. The basis polynomials for these models are now Chebyshev polynomials, which changes the notion of convolution. The space shift also changes the associated eigenfunctions and thus the corresponding Fourier transforms. In the finite case, it turns out that the Fourier transforms are the well-known discrete cosine and sine transforms (DCTs and DSTs), which were originally derived using statistical SP [5].

The described approach can also be used in 2-D SP, as demonstrated in [6] for a hexagonal space shift and in [7] for a quincunx shift.

Contribution. We derive a 1-D SP framework, called discrete-nearest-neighbor (discrete-NN) SP, that is based on the nearest-neighbor shift shown in the last column of Table I. As indicated in the table, the basis polynomials now are orthogonal polynomials on a real line. Background for these polynomials is provided in Appendix A. Discrete-space SP is a special case, but discrete-NN SP extends far beyond. As we demonstrate, discrete-NN SP is equivalent to assuming that signals reside on a weighted line graph; directed and undirected graphs are possible. We provide a set of fundamental concepts for discrete-NN SP, including the notions of z-transform, convolution, spectrum, frequency response, and Fourier transforms for both infinite and finite cases.

The platform for our work is the ASP theory, an axiomatic approach to and a generalization of linear SP [1], [3], [8], [9]. This paper extends and completes the preliminary discussion of discrete-NN shifts in [8] (where they were called generic nearest neighbor shifts).

Related work. Orthogonal polynomials have been previously used in signal analysis and processing, primarily as suitable bases for signal representation. Laguerre polynomials have been proposed for use in representation of exponentially decaying signals and speech coding [10], [11]. Sampled Legendre polynomials have been used for numerical approximation in climate modeling [12], [13]. Bases of continuous and discretized Hermite polynomials have been proposed for the electrophysiological signal compression [14]–[18], and image

This work was supported in part by NSF grant CCF-0634967. A. Sandryhaila is with the Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA. Email: asandryh@andrew.cmu.edu. J. Kovačević is with the Departments of Biomedical Engineering and Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA. Email: jelenak@cmu.edu. M. Püschel is with the Department of Computer Science, ETH Zurich, Switzerland. Email: pueschel@inf.ethz.ch.

[1] It is called *space* since the shift operator is non-directional.

TABLE I
ASP ENABLES THE DERIVATION OF DIFFERENT SP FRAMEWORKS BY CHANGING THE NOTION OF SHIFT. THE STANDARD TIME SHIFT YIELDS TRADITIONAL TIME SP, AND THE SPACE SHIFT YIELDS SPACE SP ASSOCIATED WITH THE DCTS AND DSTS. IN THIS PAPER, WE CONSIDER THE NEAREST-NEIGHBOR SHIFT.

Concept	Discrete-time SP [1]	Discrete-space SP [3]	Discrete-NN SP (this paper)
Shift:			
visualization	$p_k \xrightarrow{1} p_{k+1}$	$p_{k-1} \xleftarrow{\frac{1}{2}} p_k \xrightarrow{\frac{1}{2}} p_{k+1}$	$p_{k-1} \xleftarrow{a_{k-1}} p_k \xrightarrow{c_k} p_{k+1}$, $p_k \xrightarrow{b_k} p_k$
as operator x	$xp_k = p_{k+1}$	$xp_k = \frac{1}{2}p_{k-1} + \frac{1}{2}p_{k+1}$	$xp_k = a_{k-1}p_{k-1} + b_kp_k + c_kp_{k+1}$
Basis polynomials	$p_k(x) = x^k$ $(x = z^{-1})$	$p_k(x) = C_k(x)$ (Chebyshev polynomials)	$p_k(x) = P_k(x)$ (orthogonal polynomials)
"z-transform"	$(s_k) \to \sum s_k x^k$	$(s_k) \to \sum s_k C_k(x)$	$(s_k) \to \sum s_k P_k(x)$
Fourier transform (infinite)	DTFT	Discrete-space Fourier transforms	Section III
Fourier transform (finite)	DFT	DCTs/DSTs	Section IV

processing [19]–[21]. The analysis of birth-death processes and queueing theory have also been interpreted in terms of the associated orthogonal polynomials [22], [23]. However, to the best of our knowledge, there is no general framework for the use of orthogonal polynomials in SP. In this paper we provide such a framework and show that it is equivalent to standard time SP but based on a changed notion of shift operation. Using this insight we derive a complete set of associated basic SP concepts and show that the expansion into orthogonal polynomials is equivalent to the Fourier transform in this framework.

As mentioned above, discrete-space SP [3] is a special case of discrete-NN SP based on specific orthogonal polynomials called Chebyshev polynomials. The associated SP framework has been derived and used for the construction of fast algorithms for DCTs and DSTs [9], [24].

Finally, a connection between Gauss-Markov random fields and signal processing based on NN shifts has been identified in [8], [25], [26].

II. ALGEBRAIC SIGNAL PROCESSING THEORY

The ASP theory [1], [3], [8] is both a generalization of and an axiomatic approach to the standard linear signal processing theory. ASP is based on the concept of a signal model defined as a triple $(\mathcal{A}, \mathcal{M}, \Phi)$, where $\mathcal{A}$ and $\mathcal{M}$ are, respectively, filter and signal space, and Φ is a generalization of the z-transform. Each signal model has its own notion of the shift, filtering or convolution, the z-transform, the Fourier transform, and other concepts.

In this section, we discuss the main ASP concepts for 1-D SP and, as examples, demonstrate their instantiations for traditional infinite and finite discrete-time SP. We also introduce several terms and concepts that have not been defined before, so this section is both a review of and a complement to the ASP theory discussed in [1], [3], [8].

A. *Infinite Discrete Models*

Signal model. Infinite discrete signals are typically represented as sequences of numbers from some vector space $\mathcal{V}$: $\mathbf{s} = \left(s_k\right)_{k\in\mathbb{Z}} \in \mathcal{V} \subseteq \mathbb{C}^{\mathbb{Z}}$. The purpose of a signal model is to formally assign to $\mathcal{V}$ a signal space and a filter space and to define the notion of filtering.

The common assumptions underlying (linear) SP [2] allow filters to be connected serially and in parallel, and to be amplified. These operations satisfy distributivity laws and other properties. If we call these operations, respectively, multiplication, addition, and scalar multiplication, then the filter space $\mathcal{A}$ becomes simultaneously a ring and a vector space, i.e., an *algebra*. We denote its elements with h.

The signal space is also a vector space whose elements we will denote with s. The signal space permits an operation (filtering) of $\mathcal{A}$ that we also write as multiplication[2]: hs means s filtered with h. These properties make the signal space an $\mathcal{A}$-module that we denote with $\mathcal{M}$.

The third component of the signal model is a mapping Φ that maps discrete sequences $\mathbf{s} \in \mathcal{V}$ to the signals $s \in \mathcal{M}$. The mapping generalizes the concept of a z-transform. Together, $\mathcal{A}$, $\mathcal{M}$, and Φ form the signal model:

Definition 1 *A* signal model *for a vector space $\mathcal{V}$ is a triple $(\mathcal{A}, \mathcal{M}, \Phi)$, where $\mathcal{A}$ is a filter algebra, $\mathcal{M}$ is an associated $\mathcal{A}$-module of signals, and Φ is a bijective mapping from $\mathcal{V}$ to $\mathcal{M}$.*

The signal model is best understood by considering the example of discrete-time SP. In this case, the signals and filters are represented by finite-energy and finite-power sequences,

[2]In mathematically terms, despite being called addition, multiplication, and scalar multiplication, these operations may be instantiated in various ways, not necessarily through the traditional addition and multiplication. However, all signal models considered in this paper, have filters $h = h(x)$ and signals $s = s(x)$ represented by series or polynomials in x. For all models, the above operations are instantiated as the addition and multiplication of series or polynomials.

$$\cdots \longrightarrow \underset{x^{-1}}{\bullet} \longrightarrow \underset{x^{0}}{\bullet} \longrightarrow \underset{x^{1}}{\bullet} \longrightarrow \underset{x^{2}}{\bullet} \longrightarrow \cdots$$

Fig. 1. Visualization of the 1-D infinite discrete-time model. The weight on each edge is 1.

respectively:[3]

$$\begin{aligned}
&\mathcal{A} = \{h = \textstyle\sum_{m\in\mathbb{Z}} h_m x^m \mid \mathbf{h} = \big(\dots, h_0, h_1, \dots\big)^T \in \ell^1(\mathbb{Z})\},\\
&\mathcal{M} = \{s = \textstyle\sum_{k\in\mathbb{Z}} s_k x^k \mid \mathbf{s} = \big(\dots, s_0, s_1, \dots\big)^T \in \ell^2(\mathbb{Z})\},\\
&\Phi : \ell^2(\mathbb{Z}) \to \mathcal{M},\ \mathbf{s} \to s = \textstyle\sum_{k\in\mathbb{Z}} s_k x^k.
\end{aligned} \tag{1}$$

Hence, Φ is the standard z-transform (we substitute $x = z^{-1}$), and $\mathcal{A}$ and $\mathcal{M}$ are defined in the z-domain. Its elemnts are primarily viewed as series (infinite polynomials) rather than as functions.

The motivation for the signal model definition is that all other basic SP concepts can be derived from it as explained below. But first we briefly discuss how the discrete-time signal model (1) can be derived from the discrete-time shift operator [1]. This will later allow us to derive the discrete-NN model from the NN shift analogously.

Consider discrete-time points $p_k,\ k \in \mathbb{Z}$. The time shift moves p_k to p_{k+1} (see the second column of Table I). Calling the shift x, this can be written as an operation

$$x p_k = p_{k+1}, \quad k \in \mathbb{Z}.$$

Assuming $p_0 = 1$, the solutions of this recurrence are polynomials $p_k = p_k(x) = x^k, \quad k \in \mathbb{Z}$. The signal space $\mathcal{M}$ now consists of linear combinations of p_k. The associated filter space is *generated* by the shift, i.e., it consists of linear combinations of k-fold shifts x^k. This yields $\mathcal{M}$ and $\mathcal{A}$ as shown in (1). The restriction to $\ell^2(\mathbb{Z})$ and $\ell^1(\mathbb{Z})$ ensures that filtering a signal in $\mathcal{M}$ yields again a signal in $\mathcal{M}$:

$$hs \in \mathcal{M}, \quad \text{for } h \in \mathcal{A},\ s \in \mathcal{M}.$$

The *visualization* of a signal model is the graph associated with the operation of the shift on the basis polynomials in $\mathcal{M}$. For the time model, the visualization is shown in Fig. 1.

Convolution. As explained above, filtering a signal $s \in \mathcal{M}$ with a filter $h \in \mathcal{A}$ is written as multiplication: $hs = t \in \mathcal{M}$. The coefficients of t are given by $\mathbf{t} = \big(\dots, t_0, t_1, \dots\big)^T = \Phi^{-1}(hs)$.

For the infinite time model (1), filtering is defined by the multiplication of $h(x) \in \mathcal{A}$ and $s(x) \in \mathcal{M}$:

$$t(x) = \sum_{k\in\mathbb{Z}} t_k x^k = h(x)s(x),$$

which in coordinate form yields the standard linear convolution $\mathbf{t} = \mathbf{h} * \mathbf{s}$, where $t_k = \sum_{m\in\mathbb{Z}} h_m s_{k-m}$.

Spectrum. The *spectrum* associated with a signal model is a collection of spectral components that are *irreducible submodules* $\mathcal{M}_\omega \leq \mathcal{M}$. These are subspaces of $\mathcal{M}$ that are closed under filtering with $h \in \mathcal{A}$. The index ω is called the *frequency*.

In all models considered in this paper, the spectral components $\mathcal{M}_\omega$ have dimension one and are thus of the form

$$\mathcal{M}_\omega = \Big\{c \cdot f_\omega(x) \mid c \in \mathbb{C}\Big\},$$

where $f_\omega(x)$ is an eigenfunction for all filters $h = h(x) = \sum_m h_m x^m \in \mathcal{A}$. In particular, for $h(x) = x$, this means

$$x f_\omega(x) = \lambda_\omega f_\omega(x)$$

for some $\lambda_\omega \in \mathbb{C}$. Thus, by linearity, for any $h(x) \in \mathcal{A}$

$$h(x) \cdot f_\omega(x) = h(\lambda_\omega) f_\omega(x). \tag{2}$$

For the infinite time model (1), the eigenfunctions are

$$f_a(x) = \sum_{k\in\mathbb{Z}} a^k x^k, \quad a \in \mathbb{C}, \tag{3}$$

since $x \cdot f_a(x) = a^{-1} f_a(x)$. Hence, for any $h(x) \in \mathcal{A}$,

$$h(x) \cdot f_a(x) = h(a^{-1}) \cdot f_a(x). \tag{4}$$

Fourier transform. The Fourier transform for a signal model is constructed by projecting a signal onto the spectral components $\mathcal{M}_\omega$. It may not be necessary to use all spectral components to obtain an invertible transform.

For example, for the infinite time model (1), it is sufficient to consider only the eigenfunctions $f_a(x)$ in (3) for $|a| = 1$, i.e., those located on the unit circle, which is the interval of orthogonality of the basis functions x^k: if we parameterize $x = e^{j\omega}$, $\omega \in [0, 2\pi)$, then

$$\int_0^{2\pi} e^{j\omega k} (e^{j\omega m})^* d\omega = \int_0^{2\pi} e^{j\omega(k-m)} d\omega = 2\pi\delta_{k-m}.$$

Hence, the associated eigenfunctions (3) are[4]

$$f_\omega(x) = \sum_{k\in\mathbb{Z}} e^{j\omega k} x^k,$$

and they satisfy

$$x \cdot f_\omega(x) = e^{-j\omega} f_\omega(x). \tag{5}$$

The resulting Fourier transform is the standard discrete-time Fourier transform:

$$\begin{aligned}
S(\omega) &= \langle s, f_\omega(x)\rangle = \sum_{k\in\mathbb{Z}} s_k e^{-j\omega k},\\
s_k &= \frac{1}{2\pi} \int_0^{2\pi} S(\omega) e^{j\omega k} d\omega.
\end{aligned}$$

Frequency response. The frequency response of a filter $h \in \mathcal{A}$ is defined by its action on the spectral components $\mathcal{M}_\omega$. It is directly obtained from (2) as

$$H(\omega) = h(\lambda_\omega).$$

As follows from (4) and (5), the frequency response for the infinite time model (1) is $H(\omega) = h(e^{-j\omega}) = \sum_{m\in\mathbb{Z}} h_m e^{-j\omega m}$, $\omega \in [0, 2\pi)$.

Convolution theorem. Once the Fourier transform and

[3]Throughout this paper, spaces are denoted with calligraphic letters, abstract signals and filters (elements of $\mathcal{M}$ and $\mathcal{A}$) with italic letters, and coordinate vectors and matrices with boldface letters. The symbol $\to$ indicates a mapping.

[4]By convention, we write f_ω instead of $f_{e^{j\omega}}$

the frequency response are defined for a signal model, the convolution of a signal $s \in \mathcal{M}$ with a filter $h \in \mathcal{A}$ can be expressed via the product of their Fourier transform and frequency response (convolution theorem).

For the infinite time model (1), it follows from the definitions of the discrete-time Fourier transform and the frequency response that the convolution of $s(x) \in \mathcal{M}$ and $h(x) \in \mathcal{A}$ corresponds to the product of $S(\omega)$ and $H(\omega)$:

$$t(x) = h(x)s(x) \;\Leftrightarrow\; T(\omega) = H(\omega)S(\omega).$$

Parseval equality. The Parseval equality establishes the connection between the energy of the signal and the energy of its Fourier transform. For infinite discrete-time signals, it has the form [2]

$$\sum_{k\in\mathbb{Z}} |s_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |S(\omega)|^2 d\omega.$$

Frequency domain. We call the space of the Fourier transforms $S(\omega)$ for all $s(x) \in \mathcal{M}$ the *frequency domain*. For the infinite discrete-time model (1), the frequency domain is a Hilbert space of continuous finite-energy functions defined on $[0, 2\pi)$, with the inner product

$$\langle u, v\rangle = \int_0^{2\pi} u(\omega)v^*(\omega)d\omega.$$

The set $\left(e^{j\omega m}\right)_{m\in\mathbb{Z}}$, where $\omega \in [0, 2\pi)$, is an orthogonal basis in this frequency domain.

B. Finite Discrete Models

As demonstrated in [1], [8], 1-D linear shift-invariant models for finite discrete signals necessarily have $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$. Here, $\mathbb{C}[x]/p(x)$ denotes a polynomial algebra, which is a set of polynomials of degree less than $n = \deg(p)$ with polynomial multiplication (i.e., filtering and serial connection of filters) performed modulo $p(x)$.

Signal model. Assuming 1-D and shift-invariance, the generic signal model for finite discrete signals has the form

$$\begin{aligned} &\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x), \\ &\Phi:\ \mathbb{C}^n \to \mathcal{M},\ \mathbf{s} \to \textstyle\sum_{k=0}^{n-1} s_k p_k(x), \end{aligned} \tag{6}$$

where $b = \left(p_k(x)\right)_{0\le k<n}$ is a chosen basis for $\mathcal{M}$.

The model commonly assumed for finite discrete-time signals is

$$\begin{aligned} &\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x^n - 1), \\ &\Phi:\ \mathbb{C}^n \to \mathcal{M},\ \mathbf{s} \to \textstyle\sum_{k=0}^{n-1} s_k x^k. \end{aligned} \tag{7}$$

It can be obtained from the infinite model (1) by imposing the periodic boundary condition $x^n = x^0 = 1$, i.e., $x^n - 1 = 0$ [1], [8]. The visualization of this model, obtained from the action of the time shift x on the basis $b = \left(x^0, x^1, \ldots, x^{n-1}\right)$, is shown in Fig. 2. The periodic boundary condition is captured by the arrow that connects the boundary points of the graph.

Convolution. Filtering in the finite model (6) has the form

$$t(x) = h(x)s(x) \mod p(x). \tag{8}$$

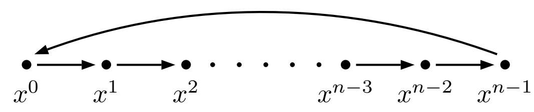


Fig. 2. Visualization of the 1-D finite discrete-time model.

In the case of the finite time model (7), this means

$$t(x) = \sum_{k=0}^{n-1} t_k x^k = h(x)s(x) \mod (x^n - 1),$$

which, in the coordinate form, becomes the standard circular convolution $\mathbf{t} = \mathbf{h} \circledast \mathbf{s}$, so that $t_k = \sum_{0\le m<n} h_{(k-m) \bmod n} s_m$.

Spectrum. We assume that the polynomial $p(x) = (x - \alpha_0)\ldots(x - \alpha_{n-1})$ is a separable polynomial, i.e. its zeros are distinct: $\alpha_k \neq \alpha_m$ for $k \neq m$. Let $\alpha = \left(\alpha_0, \ldots, \alpha_{n-1}\right)$. The spectrum is obtained from the Chinese Remainder Theorem [27] as the decomposition of the signal module $\mathcal{M} = \mathbb{C}[x]/p(x)$ into the direct sum of irreducible submodules $\mathbb{C}[x]/(x - \alpha_0), \ldots, \mathbb{C}[x]/(x - \alpha_{n-1})$:

$$\begin{aligned} \mathbb{C}[x]/p(x) &\to \mathbb{C}[x]/(x-\alpha_0) \oplus \cdots \oplus \mathbb{C}[x]/(x-\alpha_{n-1}), \\ s(x) &\to \left(s(\alpha_0),\ s(\alpha_1),\ \ldots,\ s(\alpha_{n-1})\right)^T. \end{aligned} \tag{9}$$

Hence, the spectral components of the model (6) are given by $\mathcal{M}_k = \mathbb{C}[x]/(x - \alpha_k)$, $0 \le k < n$.

In particular, the spectrum of the finite time model (7) is given by $\mathcal{M}_k = \mathbb{C}[x]/(x - e^{-j2\pi k/n})$, $0 \le k < n$.

Fourier transform. The mapping (9) defines the Fourier transform associated with the signal model (6). With respect to the basis b in $\mathcal{M}$, it is given by the matrix

$$\begin{aligned} \mathbf{P}_{b,\alpha} &= \left[p_m(\alpha_k)\right]_{0\le k,m<n} \qquad (10) \\ &= \begin{pmatrix} p_0(\alpha_0) & p_1(\alpha_0) & \ldots & p_{n-1}(\alpha_0) \\ p_0(\alpha_1) & p_1(\alpha_1) & \ldots & p_{n-1}(\alpha_1) \\ \vdots & \vdots & & \vdots \\ p_0(\alpha_{n-1}) & p_1(\alpha_{n-1}) & \ldots & p_{n-1}(\alpha_{n-1}) \end{pmatrix}. \end{aligned}$$

This means the Fourier transform (9) of a signal $\Phi(\mathbf{s}) = s(x) = \sum_{k=0}^{n-1} s_k p_k(x) \in \mathcal{M}$ and its inverse can be computed as the matrix-vector products

$$\begin{aligned} \mathbf{S} &= \mathbf{P}_{b,\alpha} \cdot \mathbf{s}, \\ \mathbf{s} &= \mathbf{P}_{b,\alpha}^{-1} \cdot \mathbf{S}, \end{aligned} \tag{11}$$

where $\mathbf{s} = (s_0, \ldots, s_{n-1})^T$ and $\mathbf{S} = (S_0, \ldots, S_{n-1})^T$.

As follows from (10), the Fourier transform for the finite time model (7) is the well-known discrete Fourier transform (DFT):

$$\mathbf{P}_{b,\alpha} = \left[e^{-j2\pi km/n}\right]_{0\le k,m<n} = \mathbf{DFT}_n\,. \tag{12}$$

Frequency response. We obtain from (9) that the projection of $s(x) \in \mathcal{M}$ on a spectral component $\mathcal{M}_k = \mathbb{C}[x]/(x - \alpha_k)$ is the evaluation $s(\alpha_k)$, since

$$s(x) \equiv s(\alpha_k) \mod (x - \alpha_k).$$

Hence, $h(x)s(x) \equiv h(x)s(\alpha_k) \equiv h(\alpha_k)s(\alpha_k) \mod (x-\alpha_k)$, and the frequency response of a filter $h(x) \in \mathcal{A}$ at α_k is

$$H_k = h(\alpha_k). \tag{13}$$

For the finite time model (7), the frequency response of a filter $h(x) = \sum_{m=0}^{n-1} h_m x^m \in \mathcal{A}$ is hence $H_k = h(e^{-j2\pi k/n}) = \sum_{m=0}^{n-1} h_m e^{-j2\pi km/n}$ and thus takes the same form as the discrete Fourier transform of a signal.

Convolution theorem. The general convolution theorem for finite discrete signal models, proven in [1], [8], establishes that the convolution (8) can be expressed via the Fourier transform (11) and the frequency response (13) as the product of the discrete Fourier transform S_k and the frequency response H_k:

$$t(x) = h(x)s(x) \mod p(x) \;\Leftrightarrow\; T_k = H_k S_k.$$

In particular, the convolution of $s(x) \in \mathcal{M}$ and $h(x) \in \mathcal{A}$ in the finite time model (7) corresponds to

$$t(x) = h(x)s(x) \mod (x^n - 1) \;\Leftrightarrow\; T_k = h(e^{-j2\pi k/n})S_k.$$

Parseval equality. The energy of the signal $\mathbf{s}$ and its Fourier transform $\mathbf{S}$ can be calculated as $||\,\mathbf{s}\,||_2 = (\mathbf{s}^*\,\mathbf{s})^{1/2}$, and $||\,\mathbf{S}\,||_2 = (\mathbf{S}^*\,\mathbf{S})^{1/2} = (\mathbf{s}^*\,\mathbf{P}_{b,\alpha}^*\,\mathbf{P}_{b,\alpha}\,\mathbf{s})^{1/2}$.

For the finite time model (7), we obtain $||\,\mathbf{S}\,||_2 = \sqrt{n}||\,\mathbf{s}\,||_2$, since $\mathbf{P}_{b,\alpha}^*\,\mathbf{P}_{b,\alpha} = \mathbf{DFT}_n^*\,\mathbf{DFT}_n = n\,\mathbf{I}_n$.

Frequency domain. The frequency domain of the finite discrete model (6) can be viewed as the frequency domain of the infinite discrete model sampled at frequencies $\omega_k = \alpha_k$.

For example, the frequency domain of the finite time model (7) is the frequency domain of the infinite time model (1) sampled at frequencies $\omega_k = 2\pi k/n$ [2], [4]. The orthogonal basis of this domain is formed by $e^{j\omega m}$, $0 \le m < n$, sampled at ω_k: the mth basis function is $\big(1, e^{-j2\pi m/n}, \ldots, e^{-j2\pi m(n-1)/n}\big)$.

III. Infinite Discrete-NN Model

In this section, we construct the 1-D infinite discrete-NN signal model based on the discrete-NN shift. We then derive the associated SP concepts and properties for the new model, exactly in parallel with the discussion in Section II. In particular, we will see that the SP framework for infinite discrete-NN signal model contains all basic concepts, but considerably differs from the traditional infinite discrete-time SP.

We start with the discrete-NN shift shown in the fourth column of Table I. Calling the shift x and assuming that it operates on points p_k, the shift operation can be written as

$$x \cdot p_k = a_{k-1}p_{k-1} + b_k p_k + c_k p_{k+1}. \tag{14}$$

If we assume that the coefficients $a_k, b_k, c_k \in \mathbb{R}$ satisfy the condition[5] $a_k c_k > 0$ for $k \ge 0$, and $p_0 = 1$ and $p_{-1} = 0$, then the solution to the recurrence (14) is a family $p_k = p_k(x) = P_k(x)$, $k \ge 0$, of *orthogonal polynomials* [28], [29].

[5]Definitions of orthogonal polynomials may specify separate conditions for a_k and c_k [28], [29]. The condition $a_k c_k > 0$ is more convenient for our purposes in this paper; it is equivalent to conditions in other definitions, as can be shown, for example, using Theorem 1.27 in [29].

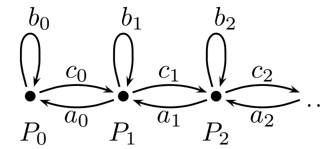


Fig. 3. Visualization of the 1-D infinite discrete-NN signal model.

These polynomials are reviewed in Appendix A. We note that alternative initial conditions on p_0 and p_{-1} are also possible; we discuss them at the end of this section.

We denote the orthogonality interval for $P_k(x)$ as $\mathcal{I} \subseteq \mathbb{R}$ and the weight function as $\mu(x)$. The orthogonality condition is

$$\int_{\mathcal{I}} P_k(x)P_m(x)\mu(x)dx = \mu_k \delta_{k-m}.$$

Signal model. Given a sequence $(P_k)_{k\ge0}$ of orthogonal polynomials that satisfy (14), we construct the *infinite discrete-NN signal model* from the NN shift the same way we constructed the infinite discrete-time model (1) from the time shift, but for infinite right-sided signals only (since $P_k(x)$ are defined only for $k \ge 0$):

$$\begin{aligned}
&\mathcal{A} = \{h = \textstyle\sum_{m\ge0} h_m x^m \mid \mathbf{h} = \big(h_0, h_1, \ldots\big) \in \ell^1(\mathbb{N}_0)\},\\
&\mathcal{M} = \{s = \textstyle\sum_{k\ge0} s_k P_k(x) \mid \mathbf{s} = \big(s_0, s_1, \ldots\big) \in \ell^2_\phi(\mathbb{N}_0)\},\\
&\Phi:\ \ell^2_\phi(\mathbb{N}_0) \to \mathcal{M},\ \mathbf{s} \to \textstyle\sum_{k\ge0} s_k P_k(x).
\end{aligned} \tag{15}$$

Here, $\mathbb{N}_0$ is the set of non-negative integers; and

$$\ell^2_\phi(\mathbb{N}_0) = \{\mathbf{s} \mid \mathbf{s} \in \ell^2(\mathbb{N}_0) \text{ and } \phi(x)\,\mathbf{s} \in \ell^2(\mathbb{N}_0)\} \subseteq \ell^2(\mathbb{N}_0)$$

is the vector space of finite-energy signals that preserve the finite-energy property when multiplied by the matrix

$$\phi(x) = \begin{pmatrix} b_0 & a_0 & & \\ c_0 & b_1 & a_1 & \\ & c_1 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{16}$$

$\phi(x)$ is called the *matrix representation of* x and satisfies [8]

$$\Phi(\phi(x)\,\mathbf{s}) = x \cdot s(x). \tag{17}$$

By the Cauchy–Schwarz inequality [30], $\ell^2_\phi(\mathbb{N}_0)$ is closed under addition and scalar multiplication; hence, it is a vector space. Furthermore, by induction, any $\mathbf{s} \in \ell^2_\phi(\mathbb{N}_0)$ satisfies $\phi^k(x)\,\mathbf{s} \in \ell^2_\phi(\mathbb{N}_0)$ for any $k \ge 1$. Then it follows from (17) that for any $s(x) \in \mathcal{M}$ and $h(x) \in \mathcal{A}$, the product $h(x)s(x) \in \mathcal{M}$, and hence $\mathcal{M}$ is an $\mathcal{A}$-module as required.

The visualization of the model is shown in Fig. 3. It starts with $P_0(x)$ and has the left boundary condition $P_{-1}(x) = 0$.

Note that here we choose $x^0, x^1, x^2, \ldots$ as a basis of the filter algebra $\mathcal{A}$. Other choices are also possible. For example, if $P_k(x)$ are Chebyshev polynomials, it is more convenient to use Chebyshev polynomials of the first kind $T_m(x)$ as a basis of $\mathcal{A}$ [3], [8].

Convolution. Filtering in the infinite NN model (15) is defined as the product of a signal $s(x) \in \mathcal{M}$ with a filter

$h(x) \in \mathcal{A}$:

$$t(x) = \sum_{k \geq 0} t_k P_k(x) = h(x)s(x).$$

Unfortunately, no general closed-form expression exists that directly describes t_k via s_k and h_m. The coefficients t_k can be obtained either from a direct expansion of $h(x)s(x)$ into the basis of $P_k(x)$, or using the convolution theorem discussed below.

Spectrum. As discussed in Section II-A, to derive the appropriate Fourier transform for the infinite NN model, we first need to identify the spectrum of $\mathcal{M}$, i.e., the eigenfunctions f_ω under the discrete-NN shift.

Theorem 1 *Let*

$$\eta_k = \prod_{m=0}^{k-1} \frac{c_m}{a_m}. \tag{18}$$

The eigenfunctions for the infinite NN model have the form

$$f_\omega(x) = \sum_{k \geq 0} \eta_k P_k(\omega) P_k(x). \tag{19}$$

For any $\omega \in \mathbb{R}$, they satisfy

$$x \cdot f_\omega(x) = \omega \cdot f_\omega(x) \tag{20}$$

Proof: For convenience, we set $c_{-1} = 0$. As follows from (14) and (17), any signal $\sum_{k\geq 0} s_k P_k(x) \in \mathcal{M}$ satisfies

$$x \cdot \sum_{k \geq 0} s_k P_k(x) = \sum_{k \geq 0} \Big(c_{k-1} s_{k-1} + b_k s_k + a_k s_{k+1} \Big) P_k(x).$$

Applying this to $f_\omega(x)$ in (19) yields

$$\begin{aligned} x \cdot f_\omega(x) &= \sum_{k \geq 0} \Big(c_{k-1} \eta_{k-1} P_{k-1}(\omega) + b_k \eta_k P_k(\omega) \\ &\quad + a_k \eta_{k+1} P_{k+1}(\omega) \Big) P_k(x) \\ &= \sum_{k \geq 0} \Big(a_{k-1} P_{k-1}(\omega) + b_k P_k(\omega) \\ &\quad + c_k P_{k+1}(\omega) \Big) \eta_k P_k(x) \\ &= \sum_{k \geq 0} \Big(\omega \cdot P_k(\omega) \Big) \eta_k P_k(x) \\ &= \omega \cdot f_\omega(x). \end{aligned}$$

■

For the spectrum we only select a subset of eigenfunctions sufficient to obtain an invertible transform. For the infinite NN model (15), it is sufficient to consider only frequencies ω from the interval of orthogonality $\mathcal{I}$, since the orthogonal polynomials $P_k(x)$ form an orthogonal basis over $\mathcal{I}$, as discussed in Appendix A.

Hence, the spectral components, indexed by $\omega \in \mathcal{I}$, are

$$\mathcal{M}_\omega = \{ c \cdot \sum_{k \geq 0} \eta_k P_k(\omega) P_k(x) \mid c \in \mathbb{C} \}.$$

Fourier transform. The Fourier transform for the infinite NN model (15) projects a signal $s(x)$ onto the eigenfunctions $f_\omega(x)$:

$$S(\omega) = \langle s(x), f_\omega(x) \rangle = \sum_{k \geq 0} s_k \eta_k P_k(\omega), \tag{21}$$

where $\omega \in \mathcal{I}$. We call (21) the *discrete-NN Fourier transform*. The corresponding inverse transform is

$$s_k = \frac{1}{\mu_k \eta_k} \int_{\mathcal{I}} S(\omega) P_k(\omega) \mu(\omega) d\omega. \tag{22}$$

It follows from the equality

$$\begin{aligned} &\frac{1}{\mu_k \eta_k} \int_{\mathcal{I}} S(\omega) P_k(\omega) \mu(\omega) d\omega \\ &= \frac{1}{\mu_k \eta_k} \int_{\mathcal{I}} \Big(\sum_{m \geq 0} s_m \eta_m P_m(\omega) \Big) P_k(\omega) \mu(\omega) d\omega \\ &= \frac{1}{\mu_k \eta_k} \sum_{m \geq 0} s_m \Big(\int_{\mathcal{I}} P_m(\omega) P_k(\omega) \eta_m \mu(\omega) d\omega \Big) \\ &= \frac{1}{\mu_k \eta_k} s_k \mu_k \eta_k = s_k. \end{aligned}$$

Frequency response. As follows from (20) and the linearity of filtering, any filter $h(x) \in \mathcal{A}$ satisfies

$$h(x) f_\omega(x) = h(\omega) f_\omega(x).$$

Hence, the frequency response of $h(x) \in \mathcal{A}$ in the infinite NN model (15) is

$$H(\omega) = h(\omega) = \sum_{m \geq 0} h_m \omega^m. \tag{23}$$

Convolution theorem. The convolution theorem for the discrete-NN model follows from the definitions of the discrete-NN Fourier transform (21) and the frequency response (23).

Theorem 2 *Given the filtered signal $t(x) = h(x)s(x)$ and the original signal $s(x)$, define scaled signals $\hat{t}(x) = \sum_{k\geq 0} \hat{t}_k P_k(x)$ and $\hat{s}(x) = \sum_{k \geq 0} \hat{s}_k P_k(x)$, where $\hat{t}_k = t_k/\eta_k$ and $\hat{s}_k = s_k/\eta_k$. Then, their discrete-NN Fourier transforms $\hat{T}(\omega) = \sum_k \hat{t}_k \eta_k P_k(\omega)$ and $\hat{S}(\omega) = \sum_k \hat{s}_k \eta_k P_k(\omega)$ satisfy*

$$t(x) = h(x)s(x) \;\Leftrightarrow\; \hat{T}(\omega) = H(\omega)\hat{S}(\omega).$$

Parseval equality. The Parseval equality for the infinite discrete-NN signal model takes the following form.

Theorem 3 *A signal $s(x) = \sum_{k \geq 0} s_k P_k(x)$ and its discrete-NN Fourier transform $S(\omega)$ satisfy the property*

$$\sum_{k \geq 0} |s_k|^2 \mu_k \eta_k^2 = \int_{\mathcal{I}} |S(\omega)|^2 \mu(\omega) d\omega. \tag{24}$$

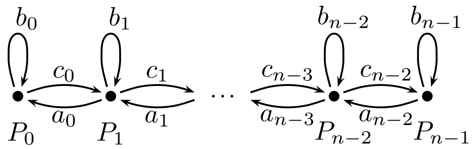


Fig. 4. Visualization of the finite discrete-NN signal model.

Proof: The Parseval equality (24) follows from

$$\begin{aligned}\int_{\mathcal{I}} |S(\omega)|^2 \mu(\omega) d\omega &= \int_{\mathcal{I}} S(\omega) S^*(\omega) \mu(\omega) d\omega \\ &= \int_{\mathcal{I}} S(\omega) \Big(\sum_{k\geq 0} s_k^* \eta_k P_k(\omega) \Big) \mu(\omega) d\omega \\ &= \sum_{k\geq 0} s_k^* \eta_k \int_{\mathcal{I}} S(\omega) P_k(\omega) \mu(\omega) d\omega \\ &= \sum_{k\geq 0} s_k^* \eta_k \mu_k \eta_k s_k = \sum_{k\geq 0} |s_k|^2 \mu_k \eta_k^2. \end{aligned}$$

■

Frequency domain. The frequency domain for the infinite NN model is the Hilbert space of all polynomials defined on the interval $\mathcal{I}$, with the inner product

$$\langle u, v \rangle = \int_{\mathcal{I}} u(\omega) v(\omega) \mu(\omega) d\omega. \tag{25}$$

The orthogonal polynomials $P_k(\omega)$, $k \geq 0$, form an orthogonal basis of this domain.

Alternative infinite NN models. Infinite NN models can be generalized by allowing more general left boundary conditions $P_{-1}(x) = bP_0(x)$ or $P_{-1}(x) = cP_1(x)$ instead of $P_{-1} = 0$, where $b \in \mathbb{R}$ and c satisfies the condition $a_0(c_0 + c) > 0$. The appropriate model (15) can be constructed by replacing b_0 with $b_0 + b$ or c_0 with $c_0 + c$, respectively, in the recurrence (14).

IV. Finite Discrete-NN Model

From Section II-B any finite 1-D linear shift-invariant discrete signal model necessarily has $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$. In this section, we consider the case $p(x) = P_n(x)$ (other cases are discussed at the end of the section). The signal space $\mathcal{M}$ has a basis $b = \big(P_0(x), P_1(x), \dots, P_{n-1}(x)\big)$ of orthogonal polynomials that satisfy recursion (14). As explained in Appendix A, the polynomial $P_n(x)$ has exactly n distinct real zeros $\alpha = \big(\alpha_0, \dots, \alpha_{n-1}\big)$, and they all lie inside the interval of orthogonality $\mathcal{I}$.

Signal model. We define the *finite discrete-NN signal model* with the right boundary condition $P_n(x) = 0$ as

$$\begin{aligned} &\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/P_n(x), \\ &\Phi: \ \mathbb{C}^n \to \mathcal{M}, \ \mathbf{s} \to s(x) = \textstyle\sum_{0\leq k<n} s_k P_k(x). \end{aligned} \tag{26}$$

This model is obtained from the corresponding infinite NN model (15) by imposing the zero boundary condition $P_n(x) = 0$. The visualization of the model is shown in Fig. 4. The right boundary condition is indicated by the absence of corresponding edges at the right boundary point.

Convolution. Filtering in the finite NN model (26) is defined as the product of $s(x)$ and $h(x)$ modulo $P_n(x)$:

$$t(x) = \sum_{k=0}^{n-1} t_k P_k(x) = h(x) s(x) \mod P_n(x). \tag{27}$$

As in Section III, there is no general closed-form expression to describe t_k via s_k and h_m. The coefficients t_k can be found either directly from the product $h(x)s(x) \mod P_n(x)$ or using the convolution theorem discussed below.

Spectrum. From Section II-B, the spectrum of the finite discrete-NN model is the set of spectral components

$$\mathcal{M}_k = \mathbb{C}[x]/(x - \alpha_k),$$

where α_k, $0 \leq k < n$, are roots of $P_n(x)$.

Fourier transform. From Section II-B, the Fourier transform for a finite discrete model is described by the matrix $\mathbf{P}_{b,\alpha}$ in (10), where $b = \big(P_0(x), \dots, P_{n-1}(x)\big)$ is the basis of $\mathcal{M}$ in (26). For the finite discrete-NN model, we call the corresponding $\mathbf{P}_{b,\alpha}$ the *discrete-NN transform*.

This transform can be easily orthogonalized, as stated in the following theorem.

Theorem 4 *Consider the diagonal matrices*

$$\begin{aligned} \mathbf{D} &= c_{n-1}\eta_{n-1} \cdot \operatorname{diag}\left(P_{n-1}(\alpha_k) P_n'(\alpha_k)\right)_{0\leq k<n}, \\ \mathbf{E} &= \operatorname{diag}\left(\eta_k\right)_{0\leq k<n}, \end{aligned} \tag{28}$$

where η_k is defined in (18), *and $P_n'(x)$ denotes the derivative of $P_n(x)$. Then*

$$\mathbf{D}^{-1/2}\, \mathbf{P}_{b,\alpha}\, \mathbf{E}^{1/2} \tag{29}$$

is an orthogonal matrix.

Proof: This result can be obtained from Theorem 16 in [8] using the following two facts. First, as shown in Appendix A, the weights μ_k can be expressed as

$$\mu_k = \mu_0 \cdot \prod_{i=0}^{k-1} \frac{a_i}{c_i} = \frac{\mu_0}{\eta_k}, \tag{30}$$

where η_k is defined in (18).

Second, the recurrence (14) for polynomials $P_k(x)$ can be written as

$$P_k(x) = \frac{x}{c_{k-1}} P_{k-1}(x) - \frac{b_{k-1}}{c_{k-1}} P_{k-1}(x) - \frac{a_{k-2}}{c_{k-1}} P_{k-2}(x).$$

If β_n denotes the leading coefficient of $P_n(x)$, it follows that $\beta_{n-1}/\beta_n = c_{n-1}$. Hence,

$$\frac{\beta_{n-1}}{\beta_n \mu_{n-1}} = \frac{\mu_0}{c_{n-1}\eta_{n-1}}. \tag{31}$$

By using (30) and (31) in the corresponding places of Theorem 16 in [8], we conclude that (29) is indeed an orthogonal matrix[6]. ■

Frequency response. The frequency response of a filter

[6] In this paper, we use a notation and indexing that is slightly different from the one in [8].

$h(x) = \sum_{m=0}^{n-1} h_m x^m \in \mathcal{A}$, given by (13), is

$$H_k = h(\alpha_k) = \sum_{m=0}^{n-1} h_m \alpha_k^m.$$

Convolution theorem. As follows from Section II-B, the convolution (27) satisfies

$$t(x) = h(x)s(x) \mod P_n(x) \;\Leftrightarrow\; T_k = H_k S_k.$$

Hence, the coordinate vector $\mathbf{t} = \big(t_0, \ldots, t_{n-1}\big)^T$ of the filtered signal $t(x)$ can be calculated as

$$\begin{aligned} \mathbf{t} &= \mathbf{P}_{b,\alpha}^{-1} \operatorname{diag}\left(H_0, \ldots, H_{n-1}\right) \mathbf{P}_{b,\alpha}\, \mathbf{s} \\ &= \mathbf{E}\, \mathbf{P}_{b,\alpha}^T\, \mathbf{D}^{-1} \operatorname{diag}\left(H_0, \ldots, H_{n-1}\right) \mathbf{P}_{b,\alpha}\, \mathbf{s}\,. \end{aligned}$$

The matrices $\mathbf{D}$ and $\mathbf{E}$ are specified in (28).

Parseval equality. The following theorem relates the weighted ℓ_2-norms of the signal and its Fourier transform. It follows from the orthogonality property (29) of $\mathbf{P}_{b,\alpha}$ and the structure of matrices $\mathbf{D}$ and $\mathbf{E}$ in (28).

Theorem 5 *A signal* $\mathbf{s} = \big(s_0, \ldots, s_{n-1}\big)$ *and its discrete-NN transform* $\mathbf{S} = \mathbf{P}_{b,\alpha}\, \mathbf{s} = \big(S_0, \ldots, S_{n-1}\big)$ *satisfy the property*

$$\mathbf{s}^*\, \mathbf{E}^{-1}\, \mathbf{s} = \mathbf{S}^*\, \mathbf{D}^{-1}\, \mathbf{S}\,.$$

Equivalently,

$$\sum_{k=0}^{n-1} \frac{|s_k|^2}{\eta_k} = \frac{1}{c_{n-1}\eta_{n-1}} \sum_{k=0}^{n-1} \frac{|S_k|^2}{P_{n-1}(\alpha_k) P_n'(\alpha_k)}.$$

Frequency domain. The frequency domain of the finite NN model (26) is the frequency domain of the infinite NN model (15) sampled at frequencies $\omega_k = \alpha_k$. In particular, the polynomials $P_0(x), \ldots, P_{n-1}(x)$ sampled at $\alpha_0, \ldots, \alpha_{n-1}$ yield basis functions for this frequency domain; the mth basis function is

$$\big(P_m(\alpha_0), \ldots, P_m(\alpha_{n-1})\big)^T.$$

This basis is not orthogonal. To construct an orthogonal basis for the frequency domain, one can apply the Gram-Schmidt procedure to the above basis. This construction is beyond the scope of this paper and can be found, for example, in [31].

Alternative finite NN models. The finite NN model (26) can be generalized by assuming any right boundary condition $P_n(x) = \sum_{m=0}^{n-1} \gamma_m P_m(x)$. The corresponding model is

$$\begin{aligned} &\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(P_n(x) - \textstyle\sum_{m=0}^{n-1} \gamma_m P_m(x)), \\ &\Phi:\; \mathbb{C}^n \to \mathcal{M},\; \mathbf{s} \to s(x) = \textstyle\sum_{0 \le k < n} s_k P_k(x). \end{aligned} \tag{32}$$

The discrete-NN transform has the form (10), provided $P_n(x) - \sum_{m=0}^{n-1} \gamma_m P_m(x)$ is a separable polynomial with n distinct zeros $\alpha_0, \ldots, \alpha_{n-1}$.

The alternative left boundary conditions discussed in Section III can also be applied to finite NN models and change the basis of the signal module.

V. Undirected NN Models

A. Normalized Orthogonal Polynomials

In general, orthogonal polynomials $P_k(x)$ have different norms: $\mu_k \neq \mu_m$ for $k \neq m$. They can be normalized as $\mu_k^{-1/2} P_k(x)$ to have the same norm 1 for all $k \geq 0$.

Normalized polynomials also satisfy a recursion of the form (14). The following theorem establishes when $P_k(x)$ have equal norms for all $k \geq 0$ and shows how to construct normalized polynomials from any family of orthogonal polynomials.

Theorem 6 *Orthogonal polynomials* $P_k(x)$ *have the same norm* $||P_k(x)||_{2,\mu} = ||P_0(x)||_{2,\mu}$, *if they satisfy a recurrence of the form*

$$x \cdot P_k(x) = a_{k-1} P_{k-1}(x) + b_k P_k(x) + a_k P_{k+1}(x), \tag{33}$$

with $P_0(x) = 1$ *and* $P_{-1}(x) = 0$. *That is, the coefficients in* (14) *satisfy* $a_k = c_k$ *for all* $k \geq 0$.

Alternatively, if polynomials $P_k(x)$ *satisfy* (14), *then the normalized polynomials* $\tilde{P}_k(x) = \mu_k^{-1/2} P_k(x)$ *satisfy*

$$\begin{aligned} x \cdot \tilde{P}_k(x) = \sqrt{a_{k-1}c_{k-1}}\, &\tilde{P}_{k-1}(x) \\ &+ b_k \tilde{P}_k(x) + \sqrt{a_k c_k}\, \tilde{P}_{k+1}(x). \end{aligned} \tag{34}$$

Proof: Theorem 7 and (18) show that $\mu_k = \mu_0/\eta_k$. Hence, if $a_k = c_k$ for all $k \geq 0$, then $\eta_k = 1$, $\mu_k = \mu_0$, and $||P_k(x)||_{2,\mu} = ||P_0(x)||_{2,\mu}$.

Next, observe that

$$\frac{\mu_{k+1}}{\mu_k} = \frac{a_k}{c_k}.$$

Then the normalized polynomials $\tilde{P}_k(x) = \mu_k^{-1/2} P_k(x)$ satisfy the recurrence

$$\begin{aligned} &x \cdot \tilde{P}_k(x) \\ &= x \cdot \mu_k^{-1/2} P_k(x) \\ &= \mu_k^{-1/2} \Big(a_{k-1} P_{k-1}(x) + b_k P_k(x) + c_k P_{k+1}(x) \Big) \\ &= a_{k-1} \sqrt{\frac{\mu_{k-1}}{\mu_k}}\, \tilde{P}_{k-1}(x) + b_k \tilde{P}_k(x) + c_k \sqrt{\frac{\mu_{k+1}}{\mu_k}}\, \tilde{P}_{k+1}(x) \\ &= \sqrt{a_{k-1}c_{k-1}}\, \tilde{P}_{k-1}(x) + b_k \tilde{P}_k(x) + \sqrt{a_k c_k}\, \tilde{P}_{k+1}(x). \end{aligned}$$

■

B. Undirected Models

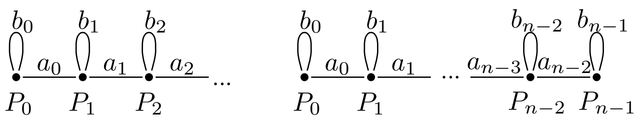
The "symmetric" NN shift (33) leads to the definition of a special class of NN models. We call them *undirected* NN models, since they are associated with undirected visualizations, i.e., signals residing on undirected graphs as shown in Fig. 5.

Theorem 6 allows us to construct infinite and finite undirected NN models. Their associated signal processing concepts have simpler forms. For example, for the undirected infinite NN model, the discrete-NN Fourier transform (21) and its inverse (22) have the form

$$\begin{aligned} S(\omega) &= \sum_{k \geq 0} s_k P_k(\omega), \\ s_k &= \frac{1}{\mu_0} \int_{\mathcal{I}} S(\omega) P_k(\omega) \mu(\omega) d\omega. \end{aligned}$$

The convolution theorem becomes

$$t(x) = h(x)s(x) \;\Leftrightarrow\; T(\omega) = H(\omega)S(\omega),$$



(a) Infinite model (b) Finite model

Fig. 5. Visualizations of undirected discrete-NN signal models.

and the Parseval equality becomes

$$\sum_{k\geq 0} s_k^2 = \frac{1}{\mu_0}\int_{\mathcal{I}} S^2(\omega)\mu(\omega)d\omega.$$

For the undirected finite NN model, the matrix $\mathbf{E}$ in the orthogonal discrete-NN transform (29) becomes an identity matrix. The Parseval equality becomes

$$\sum_{k=0}^{n-1}|s_k|^2 = \frac{1}{c_{n-1}}\sum_{k=0}^{n-1}\frac{|S_k|^2}{P_{n-1}(\alpha_k)P_n'(\alpha_k)}.$$

VI. Applications

In this section, we discuss potential applications for discrete-NN models. Some of the orthogonal polynomials considered in this section, such as Chebyshev, Hermite, and Laguerre ones, are well-known; others are constructed with the recurrence (14) using application-specific coefficients a_k, b_k, and c_k.

A. Signal representation

We could argue that the immediate motivation for novel signal models is their suitability for the description and analysis of certain classes of signals. The advantage of proposed models can be related to the improved characterization of signal properties and efficient signal processing tools they produce.

Bases of orthogonal polynomials for the representation of diverse functions have been proposed in numerous studies, some of which we discuss below. Two common applications are the representation of continuous functions using orthogonal polynomials and the representation of discrete functions using sampled orthogonal polynomials. The properties of these bases, including the calculation of projection coefficients and approximation error analysis, are discussed in [28], [29], [31]. Furthermore, for different functions, the choice of orthogonal polynomials may not be obvious and may depend on the application.

The projection coefficients in both cases can be viewed and manipulated as signals from infinite and finite discrete-NN models (15) and (26), respectively. Besides deeper insight, our framework also offers additional tools for the processing of signals based on orthogonal polynomials. For instance, since the framework defines the concepts of filtering, spectrum, and frequency response, one can potentially construct frequency-selective filters, or even filter banks, to facilitate signal analysis and processing [32].

Below, we discuss several applications that are based on the expansion of finite discrete functions into sampled orthogonal polynomials.

Electrocardiographic signal processing. We consider discrete-NN models based on Hermite polynomials $H_k(x)$. For the infinite model, the associated discrete-NN Fourier transform (21) is

$$\begin{aligned} S(\omega) &= \sum_{k\geq 0}\frac{1}{2^k k!}s_k H_k(\omega), \\ s_k &= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} S(\omega)H_k(\omega)e^{-\omega^2}d\omega. \end{aligned} \tag{35}$$

For the finite model, the discrete-NN transform (10) is orthogonalized with

$$\mathbf{D} = n\cdot \operatorname{diag}\left(H_{n-1}(\alpha_k)H_n'(\alpha_k)\right)_{0\leq k<n}, \tag{36}$$

$$\mathbf{E} = \operatorname{diag}\left(1/(2^k k!)\right)_{0\leq k<n}. \tag{37}$$

The expansion of signals into continuous and sampled Hermite polynomials has been proposed for image analysis and processing [19], [20] and electrophysiological signal compression [14]–[18]. For example, in [17], [18], we used the basis of Hermite polynomials $H_0(x),\ldots,H_{n-1}(x)$ sampled at the roots of $H_n(x)$ to efficiently represent and compress electrocardiographic signals. Analyzing the corresponding finite NN model, we concluded that sampling the electrocardiographic signals at time points proportional to the roots of Hermite polynomials is more efficient than sampling at equal time intervals. Moreover, the proposed expansion led to a significantly improved compression algorithm for electrocardiographic signals.

Speech signal analysis. Similarly to the above expansion of signals into Hermite polynomials, signals can also be expanded into Laguerre polynomials $L_k(x)$ using the corresponding discrete-NN Fourier transform

$$\begin{aligned} S(\omega) &= \sum_{k\geq 0} s_k L_k(\omega), \\ s_k &= \int_0^\infty S(\omega)L_k(\omega)e^{-\omega}d\omega, \end{aligned}$$

or the discrete-NN transform orthogonalized with $\mathbf{E}$ equal to an identity matrix and

$$\mathbf{D} = -n\cdot \operatorname{diag}\left(L_{n-1}(\alpha_k)L_n'(\alpha_k)\right)_{0\leq k<n},$$

Speech coding and a representation of exponentially-decaying signals using sampled Laguerre polynomials has been studied in [10], [11]. The analysis of the corresponding NN model may offer valuable insights for improved analysis and efficient processing of these classes of signals.

Image compression. Image compression is an extensive research area in signal processing. Multiple approaches, algorithms, and standards have been proposed for efficient image processing, compression, transmission, and storage (see [33] and references therein).

In [34], we consider the compression of multiple images with similar structures, such as faces, handwritten digits, etc. Since images are finite discrete two-dimensional signals, the corresponding NN signal model can be represented as a tensor product of two 1-D finite NN models [8], [35]. For both models, the recursion coefficients $a_k,\ b_k,\ c_k$ in (14), and hence, the corresponding orthogonal polynomials $P_k(x)$, can be obtained by solving an ℓ_2-minimization problem, and are dependent on images of interest.

The experimental results in [34] have demonstrated that the sampled orthogonal polynomials provide a suitable basis for the images of interest. In particular, a lossy image compression, implemented by preserving projection coefficients with the largest magnitudes only, achieves low error rates comparable with those for the principal component analysis and significantly lower than for other standard transforms in image coding, such as DCT and discrete wavelet transform.

Correlation analysis of Gauss-Markov random fields. Finite discrete-NN models (26) can also be relevant to the analysis of Gauss-Markov random fields [8], [25], [26]. Consider n random variables $\xi_0, \dots, \xi_{n-1}$ that satisfy the difference equation

$$\xi_k = v_{k-1}\xi_{k-1} + u_k\xi_k + v_k\xi_{k+1} + \nu_k, \tag{38}$$

where ν_k is a zero-mean Gaussian noise, and $v_k, u_k \in \mathbb{R}$ are real-valued coefficients. The set $\left(\xi_k\right)_{0 \le k < n}$ is called a *first-order Gauss-Markov random field* defined on the finite lattice $0 \le k < n$. We assume zero (Dirichlet) boundary conditions $\xi_{-1} = 0$ and $\xi_n = 0$.

The Karhunen-Loève transform (KLT), described by the eigenvector matrix of the covariance matrix $\mathbf{\Sigma}$, decorrelates the signal $\mathbf{s} = \left(\xi_0 \dots, \xi_{n-1}\right)^T$. Under certain conditions, it is considered to be the optimal transform for signal compression; however, there is no general efficient algorithm to compute this transform [36].

As demonstrated in [25], [26], the inverse of the covariance matrix $\mathbf{\Sigma}$ for the above Gauss-Markov random field is

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} u_0 & v_0 & & \\ v_0 & u_1 & \ddots & \\ & \ddots & \ddots & v_{n-2} \\ & & v_{n-2} & u_{n-1} \end{pmatrix}.$$

Let us set the values of coefficients a_k and b_k in recursion (33) to $a_k = v_k$ and $b_k = u_k$, and construct the corresponding family of normalized orthogonal polynomials $P_k(x)$. If $\mathbf{P}_{b,\alpha}$ is the discrete-NN transform for the finite NN model (26) based on the above polynomials $P_k(x)$, then $\mathbf{\Sigma}^{-1}$ can be factored as [8]

$$\mathbf{\Sigma}^{-1} = \mathbf{P}_{b,\alpha}^T\, \mathbf{D}^{-1/2} \operatorname{diag}\left(\alpha_0, \dots, \alpha_{n-1}\right) \mathbf{D}^{-1/2}\, \mathbf{P}_{b,\alpha},$$

Hence, $\mathbf{D}^{-1/2}\,\mathbf{P}_{b,\alpha}$ is precisely the KLT for the above random field. This result implies that an instantiation of the random variables $\xi_0, \dots, \xi_{n-1}$ can be viewed, analyzed, and processed as Fourier transform coefficients in the constructed finite NN model.

B. Fast signal transforms

An invertible transformation of a signal is a standard procedure in signal processing. Examples include, but are not limited to DFT, DCTs and DSTs, discrete Hartley transform, discrete Hankel transform, and KLT.

Efficient and fast implementations of these transforms is an important research problem that can be addressed using multiple approaches. One of them is based on the recognition of a transform as a polynomial transform for a finite discrete signal model (6). In this case, a decomposition of the model into a combination of simpler models corresponds to a factorization of the transform into a series of simpler transforms that may yield efficient and fast computational algorithms. The general theory of this approach has been discussed in [9], [24], [37]; early work on using polynomial transforms to derive algorithms for the DFT was done in [38].

Fast algorithms for DCTs and DSTs. In [3], [9], we demonstrated that the discrete trigonometric transforms (DCTs and DSTs) are Fourier transforms for the 1-D space model, a special kind of discrete NN-model (26) based on four kinds of Chebyshev polynomials. By exploiting the structure of the underlying signal models, in particular the polynomial algebras $\mathcal{A} = \mathcal{M}$, we derived a large class of fast algorithms, including several new ones, that require $O(n\log(n))$ additions and multiplications for a transform of size n. The derivation uses a small number of decomposition theorems for polynomial algebras.

Fast algorithms for other discrete-NN transforms. As we discussed above, discrete-NN transforms based on orthogonal polynomials different from the Chebyshev polynomials do arise and are used in various applications. Some of them are based on known orthogonal polynomials, such as Hermite or Laguerre; others are based on application-specific polynomials, such as the ones related to Gauss-Markov random fields. Overall, with the current state of knowledge, the asymptotic computational cost for a general discrete-NN transform is $O(n\log^2(n))$ operations. This cost is slightly higher than $O(n\log(n))$ operations required for the widely-used DFT, DCTs, and DSTs.

Specifically, an $O(n\log^2(n))$ algorithm was derived in [39] for discrete-NN transforms for arbitrary orthogonal polynomials, but with the right boundary condition defined by a Chebyshev polynomial. It applies to finite NN models (32) with $P_n(x) - \sum_{m=0}^{n-1}\gamma_m P_m(x)$ equal to a Chebyshev polynomial. Also, an $O(n\log^2(n)))$ algorithm for a general discrete-NN transform with an arbitrary right boundary condition was constructed in [40]. In this case, the polynomial $P_n(x) - \sum_{m=0}^{n-1}\gamma_m P_m(x)$ in (32) can be any separable polynomial. The constructed algorithm, however, can be numerically unstable. Finally, an $O(n\log(n))$ approximation algorithm was proposed in [41] for a discrete-NN transform based on Legendre polynomials with the boundary condition defined by a Chebyshev polynomial. Other work can also be found in the references of these papers. We should also mention here that algorithms requiring $O(n\log(n))$ operations are possible for discrete-NN transforms based on orthogonal polynomials other than Chebyshev ones [24], [32].

VII. Conclusions

We have proposed a new class of 1-D infinite and finite signal models based on the discrete-NN shift. We introduced relevant SP concepts for these models, including filter and signal spaces, z-transform, convolution, spectrum, Fourier transform, and frequency response. The proposed theoretical framework is fundamentally different from the traditional discrete-time SP, which is based on the discrete-time shift.

We also provided several examples of applications that inherently use the proposed signal models and may benefit from our framework. All these applications expand signals into orthogonal polynomials, which means they use the Fourier transforms associated with a suitable NN model. The proposed SP framework may lead to deeper insights and improved signal analysis and processing tools for these applications.

The computational cost of the proposed discrete-NN transforms is moderate. Algorithms for general discrete-NN transforms that require $O(n\log^2(n))$ operations are known to exist; specific cases may require only $O(n\log(n))$ operations.

Appendix A: Orthogonal Polynomials

A. Definition and Properties

There is a large body of literature dedicated to orthogonal polynomials. A thorough discussion can be found in [28], [29]. Here, we review only the properties that are used in this paper.

Definition. Polynomials $P_k(x)$, $k\geq 0$, that satisfy the three-term recurrence

$$\begin{aligned} &x\cdot P_k(x) = a_{k-1}P_{k-1}(x) + b_kP_k(x) + c_kP_{k+1}(x),\\ &P_0(x)=1,\quad P_{-1}(x)=0, \end{aligned}\tag{39}$$

where $a_k,b_k,c_k\in\mathbb{R}$ satisfy the condition $a_kc_k>0$ for $k\geq 0$, are called *orthogonal polynomials*.

Orthogonality. By Favard's theorem, there exists an interval $\mathcal{I}\subseteq\mathbb{R}$ and a weight function $\mu(x)$, non-negative on $\mathcal{I}$, such that $P_k(x)$ are orthogonal over $\mathcal{I}$ with respect to $\mu(x)$:

$$\int_{\mathcal{I}} P_k(x)P_m(x)\mu(x)dx = \mu_k\delta_{k-m}.\tag{40}$$

Here,

$$||P_k(x)||_{2,\mu} = \Big(\langle P_k(x),P_k(x)\rangle_\mu\Big)^{1/2} = \mu_k^{1/2} < \infty$$

is the L^2_μ-norm of $P_k(x)$ induced by the inner product

$$\langle f(x),g(x)\rangle_\mu = \int_{x\in\mathcal{I}} f(x)g(x)\mu(x)dx.\tag{41}$$

Basis of orthogonal polynomials. The orthogonal polynomials $P_k(x)$ form an orthogonal basis in the Hilbert space of polynomials defined on the interval $\mathcal{I}$ with the inner product (41).

Roots of $\boldsymbol{P_k(x)}$. The polynomial $P_k(x)$ has exactly k real, simple roots $\alpha_0<\ldots<\alpha_{k-1}$ that lie within the interval of orthogonality $\mathcal{I}:\alpha_i\in\mathcal{I}$ for all $0\leq i<k$. Hence, $P_k(x)$ is a separable polynomial of degree k.

In addition, $P_k(x)$ is the characteristic polynomial (up to a scalar factor) of the tridiagonal matrix

$$\mathbf{T} = \begin{pmatrix} b_0 & a_0 & & & \\ c_0 & b_1 & a_1 & & \\ & c_1 & b_2 & \ddots & \\ & & \ddots & \ddots & a_{k-2} \\ & & & c_{k-2} & b_{k-1} \end{pmatrix}.\tag{42}$$

The roots $\alpha_0,\ldots,\alpha_{k-1}$ are exactly the eigenvalues of $\mathbf{T}$. This property can be used to compute the roots of $P_k(x)$.

B. Norm calculation

In order to calculate the norm $||P_k(x)||_{2,\mu} = \mu_k^{1/2}$, we need to know the weight function $\mu(x)$ and the orthogonality interval $\mathcal{I}$. However, it may not be trivial to obtain $\mu(x)$ and $\mathcal{I}$ directly from the relation (39). Fortunately, the norm of $P_k(x)$ can be determined from the coefficients a_k, b_k, and c_k, as described by the following theorem.

Theorem 7 *The L^2_μ-norm of the polynomials $P_k(x)$ that satisfy* (14) *and are orthogonal on $\mathcal{I}$ with respect to the inner product* (41)*, is*

$$||P_k(x)||_{2,\mu} = \mu_k^{1/2} = \mu_0^{1/2}\cdot\sqrt{\prod_{i=0}^{k-1}\frac{a_i}{c_i}}.\tag{43}$$

Proof: The $k\times k$ diagonal matrix

$$\mathbf{U} = \mathrm{diag}\left(1,\sqrt{\frac{a_0}{c_0}},\sqrt{\frac{a_0a_1}{c_0c_1}},\ldots,\sqrt{\frac{a_0\ldots a_{k-2}}{c_0\ldots c_{k-2}}}\right)$$

conjugates the matrix $\mathbf{T}$ in (42) to the symmetric tridiagonal matrix

$$\mathbf{U}\,\mathbf{T}\,\mathbf{U}^{-1} = \begin{pmatrix} b_0 & \sqrt{a_0c_0} & & \\ \sqrt{a_0c_0} & b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{a_{k-2}c_{k-2}} \\ & & \sqrt{a_{k-2}c_{k-2}} & b_{k-1} \end{pmatrix}.$$

On the other hand, it was shown in [8] using the Christoffel-Darboux formula that the diagonal matrix

$$\mathbf{U}' = \mathrm{diag}\left(\mu_0^{1/2},\mu_1^{1/2},\ldots,\mu_{k-1}^{1/2}\right)$$

also conjugates $\mathbf{T}$ to a symmetric tridiagonal matrix.

Since there exists a unique (up to a constant factor) diagonal matrix that conjugates a tridiagonal matrix to a symmetric tridiagonal matrix, we conclude that $\mathbf{U}=c\,\mathbf{U}'$, and hence

$$\mu_k^{1/2} = c\cdot\sqrt{\prod_{i=0}^{k-1}\frac{a_i}{c_i}}$$

for some non-zero constant $c\in\mathbb{R}$. In particular, for $k=0$ we obtain $c=\mu_0^{1/2}$, where

$$\mu_0 = \int_{x\in\mathcal{I}}\mu(x)dx.$$

■

The following is an immediate consequence of Theorem 7:

Corollary 1 *If $a_k = c_k$ for $k \geq 0$, then $\mu_k = \mu_0$, and all $P_k(x)$ have the same norm*

$$||P_k(x)||_{2,\mu} = \sqrt{\mu_0} = ||P_0(x)||_{2,\mu}.$$

C. Classic Orthogonal Polynomials

The orthogonality interval $\mathcal{I}$ can be scaled and shifted to $[-1, 1]$, $[0, \infty)$, or $\mathbb{R}$. Based on this property, orthogonal polynomials are traditionally separated into three classes.

1) *Jacobi-like polynomials* have the orthogonality interval $[-1, 1]$. The corresponding weight function, parameterized by $a > -1, b > 1$, has the form $\mu^{(a,b)}(x) = (1-x)^a(1+x)^b$. The four kinds of Chebyshev polynomials $C_k(x) \in \{T_k(x), U_k(x), V_k(x), W_k(x)\}$ are examples of this class. They satisfy the recurrence

$$x \cdot C_k(x) = \frac{1}{2}C_{k-1}(x) + \frac{1}{2}C_{k+1}(x),$$

with boundary conditions $T_{-1}(x) = x$, $U_{-1}(x) = 0$, $V_{-1}(x) = 1$, $W_{-1}(x) = -1$. The corresponding weight functions are $\mu^{(-1/2,-1/2)}(x)$, $\mu^{(1/2,1/2)}(x)$, $\mu^{(-1/2,1/2)}(x)$, and $\mu^{(1/2,-1/2)}(x)$.

2) *Laguerre-like polynomials* have the orthogonality interval $[0, \infty)$. The corresponding weight function, parameterized by $a \in \mathbb{R}$, has the form $\mu^{(a)}(x) = x^a e^{-x}$. Laguerre polynomials $L_k(x)$ are an example of this class. They satisfy

$$x \cdot L_k(x) = -kL_{k-1}(x) + (2k+1)L_k(x) - (k+1)L_{k+1}(x)$$

and have the weight function $\mu^{(0)}(x) = e^{-x}$.

3) *Hermite-like polynomials* have the orthogonality interval $\mathbb{R}$. Their weight function, parameterized by $a > 0$, has the form $\mu^{(a)}(x) = e^{-x^2/2a}$. Hermite polynomials $H_k(x)$ are an example of this class. They satisfy

$$x \cdot H_k(x) = kH_{k-1}(x) + \frac{1}{2}H_{k+1}(x)$$

and have the weight function $\mu^{(1/2)}(x) = e^{-x^2}$.

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Aliaksei Sandryhaila (S'06–M'10) received a B.S. degree in Computer Science from Drexel University, Philadelphia, PA, in 2005, and a Ph.D. degree in Electrical and Computer Engineering from Carnegie Mellon University (CMU), Pittsburgh, PA, in 2010. He is currently a Postdoctoral Researcher in the Department of Electrical and Computer Engineering at CMU, and a Senior Research Scientist at SpiralGen, Inc. His research interests include signal processing, applied mathematics, data modeling and analysis, large-scale information processing, and design of novel efficient signal processing algorithms.

Jelena Kovačević (S'88–M'91–SM'96–F'02) received the Dipl. Electr. Eng. degree from the EE Department, University of Belgrade, Yugoslavia, in 1986, and the M.S. and Ph.D. degrees from Columbia University, New York, in 1988 and 1991, respectively.

She is a Professor of Biomedical Engineering and Electrical and Computer Engineering and the Director of the Center for Bioimage Informatics, Carnegie Mellon University, Pittsburgh, PA. Her research interests include bioimaging as well as multiresolution techniques such as wavelets and frames. From 1991–2002, she was with Bell Labs, Murray Hill, NJ. She was a co-founder and Technical VP of xWaveforms, based in New York City. She was also an Adjunct Professor at Columbia University. In 2003, she joined Carnegie Mellon University. She is a coauthor (with M. Vetterli) of the book Wavelets and Subband Coding (Prentice Hall, 1995).

Dr. Kovačević coauthored a top-10 cited paper in the Journal of Applied and Computational Harmonic Analysis, and the paper for which A. Mojsilovic received the Young Author Best Paper Award. Her paper on multidimensional filter banks and wavelets (with M. Vetterli) was selected as one of the Fundamental Papers in Wavelet Theory. She received the Belgrade October Prize in 1986 and the E.I. Jury Award at Columbia University in 1991. She was the Editor-in-Chief of the IEEE Transactions on Image Processing. She served as an Associate Editor of the IEEE Transactions on Signal Processing, as a Guest Co-Editor (with I. Daubechies) of the Special Issue on Wavelets of the Proceedings of the IEEE, Guest Co-Editor (with M. Vetterli) of the Special Issue on Transform Coding of the IEEE Signal Processing Magazine, and Guest Co-Editor (with R. F. Murphy) of the Special Issue on Molecular and Cellular Bioimaging of the IEEE Signal Processing Magazine. She has been on the Editorial Boards of the Foundations and Trends in Signal Processing, SIAM book series on Computational Science and Engineering, Journal of Applied and Computational Harmonic Analysis, Journal of Fourier Analysis and Applications, and the IEEE Signal Processing Magazine. She is a regular member of the NIH Microscopic Imaging Study Section. From 2000–2002, she served as a Member-at-Large of the IEEE Signal Processing Society Board of Governors. She is the Chair of the Bio Imaging and Signal Processing Technical Committee. She was the General Chair of ISBI 06, General Co-Chair (with V. Goyal) of the DIMACS Workshop on Source Coding and Harmonic Analysis and General Co-Chair (with J. Allebach) of the Ninth IMDSP Workshop. She was a plenary/keynote speaker at the 20 Years of Wavelets 09, European Women in Mathematics 09, MIAABWorkshop 07, Statistical Signal ProcessingWorkshop 07, Wavelet Workshop 06, NORSIG 06, ICIAR 05, Fields Workshop 05, DCC 98, as well as SPIE 98.

Markus Püschel (M'99–SM'05) is a Professor of Computer Science at ETH Zurich, Switzerland. Before, he was a Professor of Electrical and Computer Engineering at Carnegie Mellon University, where he still has an adjunct status. He received his Diploma (M.Sc.) in Mathematics and his Doctorate (Ph.D.) in Computer Science, in 1995 and 1998, respectively, both from the University of Karlsruhe, Germany. From 1998-1999 he was a Postdoctoral Researcher at Mathematics and Computer Science, Drexel University. From 2000-2010 he was with Carnegie Mellon University, and since 2010 he has been with ETH Zurich. He was an Associate Editor for the IEEE Transactions on Signal Processing, the IEEE Signal Processing Letters, was a Guest Editor of the Proceedings of the IEEE and the Journal of Symbolic Computation, and served on various program committees of conferences in computing, compilers, and programming languages. He is a recipient of the Outstanding Research Award of the College of Engineering at Carnegie Mellon and the Eta Kappa Nu Award for Outstanding Teaching. He also holds the title of Privatdozent at the University of Technology, Vienna, Austria. In 2009 he cofounded SpiralGen, Inc.