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Abstract

For all digital system designs, at the chip as well as the printed circuit board level, the propagation delays due to the physical interconnect can have a significant, even dominant, impact on system performance. Timing analyzers such as Crystal and TV attempt to capture the effect of the interconnect on the delay for reliable timing verification. For MOS integrated circuits timing analyzers are often able to predict the interconnect delay in terms of a simplified model, typically an RC tree, to within 10% of a SPICE simulation often at more than 1000x the speed. These RC tree methods of step response approximation are simple to solve but are specific to low- to mid-frequency MOS integrated circuits.

With continual progress in integrated circuit processing, operating speeds and new technologies are emerging that may require interconnect models which capture "analog" detail. Digital bipolar and high speed MOS integrated systems require a more detailed analysis than the delay estimation methods used for RC tree models. In addition, verification at the printed circuit board level also requires general RLC interconnect circuit models and consideration of the finite rise time of the signals. Currently, such digital systems are simulated with SPICE, at a prohibitive cost in CPU time.

Asymptotic Waveform Evaluation (AWE) provides a generalized approach to linear RLC circuit response approximations. The RLC interconnect model may contain floating capacitors, grounded resistors, inductors and linear controlled sources. The transient portion of the response is approximated by matching the initial boundary conditions and the first $2q - 1$ moments of the exact response to a lower order $q$-pole model. For the case of an RC tree model a first order AWE approximation reduces to the RC tree methods. Higher orders of approximation are made recursively from the first order response estimate at an incremental cost in CPU time. The order of the approximation is increased until an acceptable error is measured.
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Chapter 1

Introduction

With finer feature sizes and higher signal speeds, systems that are designed to be digital may evidence aspects of analog behavior in their interconnect, which become the ultimate determinants of performance. Timing analyzers[1,2,3,4] and timing simulators[5,6] attempt to capture the effect of the interconnect on the delay to produce reliable timing verification. For many MOS circuits, timing analyzers[1,3] are often able to predict the interconnect delay with a simplified model, typically an RC tree[7], to within 10% of a SPICE[8] simulation prediction. RC trees are RC circuits with capacitors from all nodes to ground, no floating capacitors, no resistor loops and no resistors to ground. The signal delays through an RC tree are often estimated using a form of the Elmore delay[9], which provides a dominant time constant approximation for monotonic step responses.

To enable the timing verification of bipolar circuits, the interconnect model may need to include grounded resistors[10] and inductors[11] which are not compatible with RC trees. Even for MOS circuits at particularly high speeds, the effects of coupling capacitance may need to be included in the delay estimate. Particularly at the printed circuit board level, input voltage rise time can dominate the timing of a net thus precluding the use of step response approximations for delay estimation.
Moreover, for generality, a solution is required when there are nonequilibrium initial conditions so that the delays due to precharging and charge sharing effects can be predicted.

RLC circuits with nonequilibrium initial conditions may have response waveforms which are nonmonotonic. A single time constant approximation with the Elmore delay is not generally applicable for such circuits. Two time constant models have been shown to improve the accuracy[12], but they too are applied only to RC tree monotone response approximations. Asymptotic Waveform Evaluation (AWE) provides a generalized approach to waveform estimation for RLC circuits with initial conditions and nonzero input signal rise-times. The RLC circuits may contain floating capacitors, grounded resistors, inductors and linear controlled sources. The transient response of an RLC circuit is approximated by matching the initial boundary conditions and the first 2q-1 moments of the actual response to a lower order q-pole model. For the case of an RC tree driven by a step input, a first order AWE approximation is equivalent to the methods which employ Elmore’s delay expression.

The previous work of linear RC tree delay estimation is reviewed in more detail in Chapter 2. Then AWE is described in general in terms of state variable analysis in Chapter 3. The relationship between AWE and RC tree methods is shown in Chapter 4. In Chapter 5 examples of AWE are provided for a variety of interconnect circuit models. Finally, some other applications of AWE are discussed in Chapter 6 followed by our concluding remarks in Chapter 7.
Chapter 2

Background

A typical RC Tree approach to MOS timing estimation models the MOSFETs in terms of linear approximate resistors and capacitors and estimates the delays in terms of the resulting linear circuit[1,3,5,6]. The MOSFET gates are modeled as linear capacitors and the channels as linear resistors determined as functions of their process parameters and the voltage changes which are to appear across the gates. This linear model is a simple elaboration of the switch model proposed by Bryant[13]. The interconnect is modeled by RC tree networks. The MOS circuit delay is then estimated in terms of the delay through the linear approximate circuit. The more popular methods of estimating the delay through these linear RC trees will be reviewed in this chapter.

2.1 Delay Estimation for Linear RC Trees

There are many definitions of delay given the actual transient response. The most straight-forward is the time at which the output transient rises to 50% of its final value, as shown in Figure (2.1). For efficient timing analysis, a delay formula is required which does not entail the calculation of the exact response. Elmore[9] proposed an expression for approximating the time at which the transient step
response would reach 50% of its final value for monotonic waveforms.

2.1.1 The Elmore Delay

The Elmore delay arose from the need to have a measure for damped linear networks which do not require solution for the actual transient response. The waveforms are restricted to be monotonic unit step responses and initial conditions are not allowed. An expression for the timing delay \( T_D \) in Figure (2.1) is obtained in terms of the first moment of the derivative of the step response, \( v(t) \):

\[
T_D = \int_0^\infty t \dot{v} \, dt. \tag{2.1}
\]

This expression represents the first moment, or mean, of the impulse response provided that the circuit is initially in equilibrium. Referring to Figure 2.1, the mean of the impulse response is a reasonable approximation for the time from which the step response beginning at \( t = 0 \) reaches 50% of its final value at 0.5 volts.

The value of \( T_D \) is a function only of the circuit parameters. To show this, consider the normalized impulse response of a linear passive circuit.

\[
H(s) = \frac{1 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{1 + b_1 s + b_2 s^2 + \ldots + b_n s^n} \tag{2.2}
\]
By definition, $H(s)$ is given by

$$H(s) = \int_0^{\infty} \dot{v} e^{-st} \, dt.$$  \hspace{1cm} (2.3)

Expanding $e^{-st}$ into its Taylor series, the first moment[9] of $H(s)$ is equal to

$$T_D = \int_0^{\infty} t \dot{v} \, dt,$$  \hspace{1cm} (2.4)

the Elmore delay. Expanding equation (2.2) about $s = 0$ the first moment of $H(s)$ is given by $b_1 - a_1$. Therefore the Elmore delay can be written as

$$T_D = b_1 - a_1,$$  \hspace{1cm} (2.5)

a function only of circuit parameters. It is therefore by definition that the Elmore delay is not applicable to general input signals or nonequilibrium initial conditions.

The Elmore delay estimate is a single value $T_D$. Such an estimate does not consider the logic thresholds of actual MOS devices. To do so requires finding an approximating response waveform and determining the time at which the logic threshold is crossed[11]. The Elmore delay, or first moment of the impulse response, is a good approximation for the dominant time constant of the step response[11]. Thus the step response waveform may be approximated by a single exponential function with $T_D^{-1}$ as the pole.

$$v(t) \approx v(\infty)(1 - e^{-t/T_D})$$  \hspace{1cm} (2.6)

### 2.1.2 Using the Elmore Delay for Timing Analysis

Penfield and Rubenstein[7] were the first to apply the Elmore delay as a dominant time constant approximation to determine the nominal delay in linear RC tree circuits. An RC tree is a network of capacitors and resistors with a capacitor from each node to ground, no floating capacitors, no resistor loops and no resistors to ground. With an RC tree, Elmore's expression applies, since all of the response
waveforms are monotonic[14] and the steady state solution to a unit step is 1 volt for every node. The Elmore delay can be found by inspection as a summation of series path resistance and shunt capacitance values when the circuit is restricted to the configuration of an RC tree[11]. The Elmore delay expressed as a summation of R's and C's can also be bounded for worst and best case responses[7,14]. As in the original[9], only step inputs are considered and initial conditions are always assumed to be zero.

Rubenstein, Penfield and Horowitz[14] showed that the first moment of the impulse response is also equivalent to the area between the step response and the unit step. Referring to Figure (2.1), the Elmore delay \( T_D \) is equal to the shaded area between the step response and the steady state. This equivalence can be verified upon integrating equation (2.1) by parts:

\[
T_D = \int_0^\infty t \frac{dv}{dt} \, dt = tv(t) \bigg|_0^\infty - \int_0^\infty v(t) \, dt
= \int_0^\infty [v(\infty) - v(t)] \, dt - t[v(\infty) - v(t)] \bigg|_0^\infty
\]

(2.7)

The last term in (2.7) is zero, therefore

\[
T_D = \int_0^\infty [v(\infty) - v(t)] \, dt.
\]

(2.8)

### 2.1.3 Nonequilibrium Initial Conditions

With the ability to define the Elmore delay in terms of the area between the step response and the steady state, Lin and Mead[5] have extended the area definition to include nonmootone step response waveforms. Since nonmonotonic waveforms can be so handled, nonequilibrium initial conditions are admissible as well. In that the steady state \( v(\infty) \) must be known to calculate \( T_D \), the circuit configurations for this extension are restricted to RC meshes[11]. An RC mesh allows for loops of resistors in addition to the basic definition of an RC tree. So long as there are
no resistance paths to ground the unit step response steady state solution is still known to be 1 volt at every node.

2.1.4 Grounded Resistors

RC meshes are adequate for specifying a linear equivalent model of most MOS circuits, but resistance to ground may be required to model general circuit configurations. A specific example of this necessity is the case of bipolar circuits. Resistance to ground requires that the steady state be obtained and that the delay equation be scaled by the magnitude of the voltage transition. This extension has been attempted in [10,15,16] independently, all of which result in an expression similar to

\[ T_D = \frac{1}{v(\infty) - v(0)} \int_0^\infty [v(\infty) - v(t)] \, dt. \] (2.9)

2.1.5 Two-Pole Model

Arbitrary initial conditions on the RC tree node voltages can bring about a low-frequency zero which may partially cancel the dominant pole, \( T_D^{-1} \). Such a situation cannot be modeled successfully with a single time constant model. Horowitz[17,12] proposed a model with a slow and fast time constant component for this class of circuits. Horowitz approximates the capacitor voltages with a two-pole, one-zero circuit transfer function model and finds the poles and zero by matching boundary conditions and the first moment of the model response to that of the actual response. In order to determine the poles and zeros uniquely it is assumed that the sum of the two poles is equal to the sum of all the poles in the circuit.
2.2 Conclusions

These RC tree approaches to timing estimation have been used successfully for timing analysis[1][2][3] and timing simulation[5][6] of low- to mid-frequency MOS digital integrated circuits. The single time-constant and double time-constant models provide good delay estimates for RC tree paths when driven by step inputs; higher order approximations may be required, however, when inductance and floating capacitance effects are not negligible. Although nonequilibrium initial conditions are considered for the one- and two-pole models, they are valid for a limited set of conditions and may be unable to provide a means of handling the nonmonotone waveforms which may result in general.
Chapter 3

Asymptotic Waveform Evaluation

The RC tree methods described in the previous chapter work very well for modeling MOS designs at moderate frequencies. Timing analysis of more general digital circuits may require models more elaborate than RC trees driven by single step inputs. Analysis of RLC interconnect circuit models with initial conditions and non-monotone response waveforms requires a more comprehensive waveform estimator. Asymptotic Waveform Evaluation (AWE) is a generalized approach to approximating the waveform response of linear circuits with multiple step and ramp input signals and unrestricted nonequilibrium initial conditions.

3.1 The AWE Approximation

AWE is most conveniently explained in general in terms of the differential state equations for a lumped, linear, time-invariant circuit:

\[ \dot{x} = Ax + Bu, \]  

(3.1)

where \( x \) is the \( n \)-dimensional state vector and \( u \) is the \( m \)-dimensional excitation vector. In all but the most pathological cases such a circuit description exists\[18\]. Once AWE is explained in general, the results will be particularized to the more fa-
miliar RC tree circuits for comparison with the present practices of delay estimation from the previous chapter.

Suppose that the excitation is of the form

\[ u_p(t) = u_0 + u_1 t, \quad t \geq t_0, \]  

(3.2)

where \( u_0 \) and \( u_1 \) are constant \( m \)-dimensional vectors. In general the form of \( u_p(t) \) need not be confined to such simple signals, but rather could assume any form of input excitation for which a particular solution can easily be obtained. Inputs that are polynomials in time or sums of complex-valued exponentials can in theory be as easily accommodated as the step/ramp combination in expression (3.2). For present purposes this simple class of input excitations is considered because it is adequate for the investigation of delay and rise-time effects.

For the excitation \( u_p \), (3.2), the differential-state equation (3.1) has the particular solution

\[ x_p(t) = -A^{-1}Bu_0 - A^{-2}Bu_1 - A^{-1}Bu_1 t. \]  

(3.3)

The \( A \)-matrix may not be singular for this particular solution to exist. This condition is equivalent to specifying that the circuit in question have a unique and well defined dc solution when all of its capacitances are open-circuited and all of its inductances are short-circuited. This is not an unreasonable restriction for the class of circuits for which delay estimation may be of interest. To complete the solution of (3.1), leaves the homogeneous equation

\[ \dot{x}_h = Ax_h; \]  

(3.4)

now with the initial condition

\[ x_h(0) = x_0 + A^{-1}Bu_0 + A^{-2}Bu_1, \]  

(3.5)

where \( x_0 \) is the initial state at time zero. The Laplace transform solution of the homogeneous equation is
3.1. **THE AWE APPROXIMATION**

\[ X_h(s) = (sI - A)^{-1}x_h(0). \] (3.6)

To approximate this solution, \( X_h(s) \) is first expanded in a Maclaurin series

\[ X_h(s) = -A^{-1}(I + A^{-1} s + A^{-2} s^2 + \ldots) x_h(0), \] (3.7)

and as many moments as necessary or desirable are matched in terms of lower-order approximating functions. The justification for such a moment matching approach follows from the Laplace transform definition

\[ X(s) = \int_0^\infty e^{-st} x(t) dt = \sum_{k=0}^\infty \frac{1}{k!} (-s)^k \int_0^\infty t^k x(t) dt, \] (3.8)

since it has long been established that the time moments,

\[ m_k = \frac{(-1)^k}{k!} \int_0^\infty t^k x(t) dt, \] (3.9)

provide excellent measures of delays, rise-times, etc.[9,19]. Focusing for now on a specific component of \( X_h(s) \), say the \( i^{th} \) initial condition and first \( 2q - 1 \) moments (from equation 3.7) are characterized as

\[
\begin{align*}
[m_{-1}]_i & = [-x_h(0)]_i \\
[m_0]_i & = [-A^{-1} x_h(0)]_i \\
[m_1]_i & = [-A^{-2} x_h(0)]_i \\
& \vdots \\
[m_{2q-2}]_i & = [-A^{-2q+1} x_h(0)]_i
\end{align*}
\] (3.10)

The initial conditions are represented as the negative first moment since they correspond to the coefficient of the \( s^{-1} \) term for an expansion about \( s = \infty \). It is the moments and the initial conditions from expression (3.10) that are matched to a lower-order frequency domain function of the form

\[ \hat{X}_i(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \ldots + \frac{k_q}{s - p_q} = \sum_{l=1}^q \frac{k_l}{s - p_l} = -\sum_{l=1}^q \frac{k_l/p_l}{1 - s/p_l} \] (3.11)
where \( p_1 \) through \( p_q \) are the complex approximating poles and \( k_1 \) through \( k_q \) their appropriate residues. In other words, the time domain moments are to be matched to those of an approximating function of the form

\[
\dot{x}_i(t) = \sum_{l=1}^{q} k_l e^{p_l t}.
\]

Under the assumption that the moments (3.10) can be generated easily — more on this when computational considerations are discussed — what remains now is to solve for the time constants and their corresponding residues. Expanding each of the terms in (3.11) into a series about the origin, and upon inclusion of the initial conditions, the following set of nonlinear simultaneous equations for the \( i \)th state variable is obtained:

\[
\begin{align*}
-(k_1 + k_2 + \ldots + k_q) &= [m_{-1}]_i \\
-(\frac{k_1}{p_1} + \frac{k_2}{p_2} + \ldots + \frac{k_q}{p_q}) &= [m_0]_i \\
-(\frac{k_1}{p_1^2} + \frac{k_2}{p_2^2} + \ldots + \frac{k_q}{p_q^2}) &= [m_1]_i \\
&\vdots \\
-(\frac{k_1}{p_1^{2q-1}} + \frac{k_2}{p_2^{2q-1}} + \ldots + \frac{k_q}{p_q^{2q-1}}) &= [m_{2q-2}]_i
\end{align*}
\]

(3.13)

Solution for the approximating poles and residues from this set of nonlinear equations could proceed in terms of Newton-Raphson[20] or a similar iteration method. The complexity of these indirect solution methods, however, is not fixed, but varies with the problem. Moreover, heuristics are needed to monitor the iteration step size to control convergence.

Instead of attempting to solve the nonlinear equations given by 3.13, we will reformulate the problem to allow for direct solution of the approximating poles and residues. The set of equations in 3.13 can be summarized in matrix form as

\[
-\mathbf{Vk} = [m_i]
\]

(3.14)
and

\[ \mathbf{V} \Lambda^{-q} \mathbf{k} = [\mathbf{m}_h]_1, \quad (3.15) \]

where \( \mathbf{m}_l \) represents the low-order moments \((-1, 0, \ldots, q - 2)\), \( \mathbf{m}_h \) represents the high-order moments \((q - 1, q, \ldots, 2q - 2)\), \( \Lambda^{-1} \) is a diagonal matrix of the reciprocal complex poles, and \( \mathbf{V} \) is the well-known Vandermonde matrix [21]:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
p_1^{-1} & p_2^{-1} & \cdots & p_q^{-1} \\
p_1^{-2} & p_2^{-2} & \cdots & p_q^{-2} \\
\vdots & \vdots & \ddots & \vdots \\
p_1^{-q+1} & p_2^{-q+1} & \cdots & p_q^{-q+1}
\end{bmatrix}
\]

(3.16)

It follows then from (3.14) that

\[ \mathbf{k} = -\mathbf{V}^{-1} \mathbf{m}_l \quad (3.17) \]

and

\[ \mathbf{V} \Lambda^{-q} \mathbf{V}^{-1} \mathbf{m}_l = \mathbf{m}_h. \quad (3.18) \]

Since the Vandermonde matrix is the modal matrix for a system matrix in companion form[19], equation (3.18) is equivalent to

\[ \mathbf{A}_c^{-q} \mathbf{m}_l = \mathbf{m}_h, \quad (3.19) \]

where

\[
\mathbf{A}_c^{-1} =
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
- a_0 & - a_1 & - a_2 & \cdots & - a_{q-1}
\end{bmatrix}
\]

(3.20)
with the coefficients normalized so that \( a_q = 1 \). This matrix is characterized as \( \mathbf{A}_c^{-1} \) rather than \( \mathbf{A}_c \) because its eigenvalues are the reciprocals of the approximating poles for the original system (3.1). It is shown in Appendix A that the set of simultaneous nonlinear equations (3.19) for the coefficients \( a_0 \) through \( a_{q-1}, a_c \), can be rewritten recursively to yield the following set of linear equations:

\[
\begin{bmatrix}
  m_{-1} & m_0 & \ldots & m_{q-2} \\
  m_0 & m_1 & \ldots & m_{q-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  m_{q-2} & m_{q-1} & \ldots & m_{2q-3}
\end{bmatrix}
\begin{bmatrix}
  -a_0 \\
  -a_1 \\
  \vdots \\
  -a_{q-1}
\end{bmatrix}
= 
\begin{bmatrix}
  m_{q-1} \\
  m_q \\
  \vdots \\
  m_{2q-2}
\end{bmatrix}
\tag{3.21}
\]

It is in terms of \( a_c \) that we can form a characteristic polynomial

\[
a_0 + a_1 p^{-1} + a_2 p^{-2} + \ldots + a_{q-1} p^{-q+1} + p^{-q} = 0
\tag{3.22}
\]

the roots of which are the reciprocals of the desired poles. So as not to confuse the issue, it is assumed that these poles are distinct so that (3.17) for the residues has a solution.

If the poles are not distinct, the Vandermonde matrix is by definition singular. For such cases the residues must be found using an expression other than (3.17).

For the case of a double pole given a second order approximation:

\[
\hat{X}(s) = \frac{k_1}{(s - p_1)^2} + \frac{k_1}{s - p_1}.
\tag{3.23}
\]

Expanding the terms in (3.23) into a series about \( s = 0 \):

\[
\hat{X}(s) = \frac{k_1}{p_1^2} (1 + \frac{2s}{p_1} + \frac{3s^2}{p_1^2} + \frac{4s^3}{p_1^3} + \ldots) + \frac{k_1}{p_1} (1 + \frac{s}{p_1} + \frac{s^2}{p_1^2} + \ldots).
\tag{3.24}
\]

From equation (3.24) it is apparent that the poles and residues are related to the moments by
\[ \begin{bmatrix} p_1^{-2} & p_1^{-1} \\ 2p_1^{-3} & p_1^{-2} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} m_{-1} \\ m_0 \end{bmatrix} \] (3.25)

More generally, for an \( r \)-order root, the approximate residues are related to the approximate poles by

\[ \begin{bmatrix} (-1)^{r-1} & (-1)^{r-1} & \cdots & -1 \\ p^{-r} & p^{-(r-1)} & \cdots & p^{-1} \\ \frac{r!}{(r-1)!}p^{-(r+1)} & \frac{(r-1)!}{(r-2)!}p^{-(r)} & \cdots & p^{-2} \\ \frac{(r+1)!}{(r-1)!2!}p^{-(r+2)} & \frac{(r)!}{(r-2)!2!}p^{-(r+1)} & \cdots & p^{-3} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(2r-2)!}{(r-1)!2!(r-1)!}p^{-(2r-1)} & \frac{(2r-3)!}{(r-2)!2!(r-1)!}p^{-(2r-2)} & \cdots & p^{-r} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{bmatrix} = \begin{bmatrix} m_{-1} \\ m_0 \\ \vdots \\ m_{r-2} \end{bmatrix} \] (3.26)

Expression (3.21) can still be applied in cases of repeated poles to find the approximating characteristic polynomial. This expression arises also in the model order reduction problem much studied in linear control system theory[22]. Typically in control theory the model-order reduction problem is stated first in terms of a rational transfer function

\[ \hat{X}(s) = \frac{1 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{1 + a_1 s + a_2 s^2 + \ldots + a_n s^n} \] (3.27)

to which a lower-order rational transfer function approximation is sought. For the case of \( m = n - 1 \), matching the moments from the expansion of equation (3.27) to the first \( m + n + 1 \) circuit moments results in expression (3.21)[22].

To summarize, determining the set of \( q \) approximating poles and residues from the moments requires first: solving a \( q^{th} \) order set of linear equations (equation 3.21) by Gaussian elimination to find \( a \); then solving for the roots of \( a \) from equation 3.22 to determine the approximating poles; and finally determining the residues by solving the \( q \) simultaneous linear equations from expression 3.17 (or 3.26 for the repeated root case). For the low orders of approximation that are needed for the intended application of AWE, the roots of \( a \) can be obtained explicitly, and the complexity
of the solutions for equations 3.17 and 3.21 is modest at $O(q^3)$. In such cases the runtime is dominated by the calculation of the moments, which typically is $O(n^{1.8})$ with $n$ being the number of state variables.

### 3.2 Complexity

The computation of the powers of $A^{-1}$ to obtain the moments (3.10) may look to be more complicated than it actually is. In general the $A$-matrix for a lumped, linear, time invariant circuit takes the following form [18]:

$$
A = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}^{-1} \begin{bmatrix} H_{CC} & H_{CL} \\ H_{LC} & H_{LL} \end{bmatrix}
$$  \hspace{1cm} (3.28)

The $C$ and $L$ submatrixes are symmetric diagonally dominant descriptions of the capacitance and inductance portions of the circuit. If there are no capacitance-voltage source loops or no inductance-current source cutsets in the circuit, these submatrixes reduce, respectively, to diagonal. The $H$ matrix in (3.28) is merely the hybrid characterization of the dc circuit that results upon zeroing all original independent sources and forming ports appropriately for the energy storage elements[18].

It follows that $A^{-1}$ is

$$
A^{-1} = \begin{bmatrix} H_{CC} & H_{CL} \\ H_{LC} & H_{LL} \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}
$$  \hspace{1cm} (3.29)

It is advantageous in practice that the energy storage matrix need not be inverted to find $A^{-1}$ since the models which result from circuit extraction may include many parasitic energy storage elements with large variations in magnitude amongst them.

The moments are computed recursively

$$
m_{-1} = -x_h(0)
$$  \hspace{1cm} (3.30)

and
\[ m_{k+1} = A^{-1}m_k, \quad \text{for } k = (0, 1, \ldots, 2q - 2). \quad (3.31) \]

The energy storage matrix is sparse, symmetrical and easily applied. So once the H-matrix is LU-factored the major task in computing even higher moments is repeated forward- and back-substitution of these LU-factors. Even the LU-factorization is not formidable. Such an analysis must be performed in any case to obtain the steady state solution. Moreover, in the following chapter it is shown that for several interconnect circuit models, RC trees included, the LU factors need not be found at all.

### 3.3 Stability

Expression (3.21) is one of many alternative moment matching methods that may be employed to obtain the approximating poles [22,23,24,25]. AWE differs from those of control system theory in that the zeros are not found directly, but rather the residues are obtained in order to approximate the time response. More importantly, AWE differs in that the particular solution is subtracted \textit{a priori} and only the transient portion of the response is approximated. By focusing on the residues and the homogeneous response, the lower-order approximation can be forced to match the initial state, \( m_{-1} \), and the finite integral of the voltage response, \( m_0 \). Since the integral of the approximating voltage waveform is finite and equal to the exact, the final value must also match that of the exact, thus stability is guaranteed.

There are instances when the homogeneous response waveform is nonmonotone and a first order approximation, i.e. a single exponential function, cannot match the integral of the voltage, \( m_0 \). The first order AWE approximation may prove in such cases to have no solution, or may result in a positive approximating pole. These situations are easily remedied by moving to a higher order of approximation.
3.4 Accuracy

\textbf{AWE} has been described as an approach to variable accuracy linear circuit analysis, but before it can be put to practical use, a criterion is needed for determining when the approximation is "good enough". The accuracy, or the error of a waveform estimate, must be measured to determine if higher orders of approximation should be sought. In [7] the approximations were strictly first order and absolute bounds were derived for the monotone RC tree response waveforms. The \textbf{AWE} waveforms may be nonmonotone and attempts at similar bounding of the response waveforms produced results which were extremely pessimistic. In addition, for orders of approximation greater than one, determining the absolute waveform bounds appeared to be a computationally burdensome task.

\textbf{AWE}, unlike the RC tree methods, can extend the order of the approximation efficiently, such that an acceptable waveform prediction can always be attained. Therefore, with \textbf{AWE}, instead of bounding the approximate waveform, we intend to increase the order of the approximation until the desired waveform accuracy is obtained. Moving to higher orders of approximation is done efficiently, but it is necessary that the calculation of the accuracy criterion is efficient as well. A desirable criterion would be to choose the order of approximation based upon the relative values of the moments as they are calculated recursively. Unfortunately, we were unable to determine a relation between the time moments and the accuracy of the time domain response waveform. Thus in \textbf{AWE} the accuracy is measured directly from the time response waveforms, which involves first mapping the moments to the approximating poles and residues.

The accuracy measure refers to \textit{how well} the \textbf{AWE} \(q\)\textsuperscript{th} order model approximates the \(n\)\textsuperscript{th} order circuit response. For our purposes, the time domain waveform accuracy is ideally measured by the difference between the approximate response waveform and the exact response waveform over the time range of interest. Referring to Figure
3.4. ACCURACY

3.1, the accuracy is a measure of the shaded region.

![Diagram showing V(t) with exact and approximate regions shaded]

Figure 3.1: The shaded region can be used to measure the accuracy.

The error indicated in Figure 3.1 can be expressed simply in terms of the integral of the squared waveform difference,

\[ Error = \sqrt{\int_0^\infty (v_{\text{exact}}(t) - \hat{v}(t))^2 \, dt}, \]

(3.32)

where \( \hat{v}(t) \) is the \( q^{th} \) order AWE approximation. Assuming that all of the approximate poles are real,

\[ \hat{v}(t) = \sum_{i=1}^{q} \hat{k}_i e^{\hat{\rho}_i t}. \]

(3.33)

The relative error can be found by normalizing equation 3.32 by

\[ \sqrt{\int_0^\infty v_{\text{exact}}(t)^2 \, dt}. \]

(3.34)

Of course, the exact response is not available for determining the error from equation 3.32. Instead we intend to approximate the error term by replacing the exact response in equation 3.32 with the \( q + 1 \) order AWE approximation:

\[ v(t) = \sum_{i=1}^{q+1} k_i e^{\rho_i t} \]

(3.35)

The approximate normalized error expression is
\[
\text{Error} = \sqrt{\frac{\int_0^\infty \left( \sum_{i=1}^{q+1} k_i e^{\rho_i t} - \sum_{i=1}^{q} \hat{k}_i e^{\hat{\rho}_i t} \right)^2 dt}{\int_0^\infty \left( \sum_{i=1}^{q+1} k_i e^{\rho_i t} \right)^2 dt}}. \tag{3.36}
\]

Calculating the error from equation 3.36 can be computationally intensive when the order of the approximation is large. For example, when \( q \) is equal to 4, evaluating the numerator term in equation 3.36 may require more than 40 potentially complex multiplication and addition operations. To reduce the complexity we solve for an upper bound of the numerator term using Cauchy’s inequality\[26\]:
\[
\left( \sum_{i=1}^{q+1} k_i e^{\rho_i t} - \sum_{i=1}^{q} \hat{k}_i e^{\hat{\rho}_i t} \right)^2 \leq (q + 1) \sum_{i=1}^{q+1} (k_i e^{\rho_i t} - \hat{k}_i e^{\hat{\rho}_i t})^2 \tag{3.37}
\]
It follows that
\[
\int_0^\infty \left( \sum_{i=1}^{q+1} k_i e^{\rho_i t} - \sum_{i=1}^{q} \hat{k}_i e^{\hat{\rho}_i t} \right)^2 dt \leq (q + 1) \sum_{i=1}^{q+1} \int_0^\infty (k_i e^{\rho_i t} - \hat{k}_i e^{\hat{\rho}_i t})^2 dt. \tag{3.38}
\]
Cauchy’s formula is exact when the individual exponential terms of \( v(t) \) and \( \hat{v}(t) \) match up exactly. Therefore to determine the least pessimistic result from equation 3.36, the individual \( v(t) \) and \( \hat{v}(t) \) terms should be paired in equation 3.38 by the poles and residues which lie closest to one another. In addition to ordering the poles, there must also be a way to match \( q + 1 \) terms from \( v(t) \) with only \( q \) terms from \( \hat{v}(t) \). The most straightforward approach is to match the first \( q - 1 \) terms by pole and residues values as described above, leaving only the final three terms \( v_q, v_{q+1} \) and \( \hat{v}_q \). These terms can be matched as before by splitting the \( v_q \) term into two parts and evaluating
\[
\int_0^\infty (k_q e^{\rho_q t} - k_q e^{\hat{\rho}_q t})^2 dt, \tag{3.39}
\]
and
\[
\int_0^\infty (k_{q+1} e^{\rho_{q+1} t} - (k_q - k_q) e^{\hat{\rho}_q t})^2 dt. \tag{3.40}
\]
Since equation 3.37 is valid only for real functions, the integrals of the individual differences,
3.4. ACCURACY

\[ E_i = \int_0^\infty (k_i e^{\rho t} - \hat{k}_i e^{\hat{\rho} t})^2 \, dt \]  

(3.41)

must be real numbers. Equation 3.41 can be shown to result in the expression:

\[ E_i = -\frac{k_i^2}{\rho_i} - \frac{\hat{k}_i^2}{\hat{\rho}_i} + \frac{2k_i\hat{k}_i}{\rho_i + \hat{\rho}_i}. \]  

(3.42)

When the AWE approximation contains complex pole pairs, they are evaluated in pairs so that the individual term differences are real functions and Cauchy’s inequality still applies.

\[ E = \int_0^\infty (ke^{pt} + k^* e^{pt} - \hat{k}e^{\hat{p}t} - \hat{k}^* e^{\hat{p}t})^2 \, dt \]  

(3.43)

Equation 3.43 can be shown to evaluate to a real function of the poles and residues (see Appendix B):

\[ k = k_0 + k_1 j \]

\[ k^* = k_0 - k_1 j \]

\[ p = p_0 + p_1 j \]

\[ p^* = p_0 - p_1 j \]  

(3.44)

\[ E = \frac{p_0(k_0^2 - k_1^2) + 2k_0k_1p_1}{p_0^2 + p_1^2} + \frac{\hat{p}_0(\hat{k}_0^2 - \hat{k}_1^2) + 2\hat{k}_0\hat{k}_1\hat{p}_1}{\hat{p}_0^2 + \hat{p}_1^2} \]

\[ + \frac{k_0^2 + \hat{k}_0^2}{p_0^2} - 4 \frac{(k_0 \cdot \hat{k}_0 - k_1 \cdot \hat{k}_1)(p_0 + \hat{p}_0) + (k_0 \cdot \hat{k}_0 + k_1 \cdot \hat{k}_1)(p_1 + \hat{p}_1)}{(p_0 + \hat{p}_0)^2 + (p_1 + \hat{p}_1)^2} \]

\[ + \frac{k_1^2 + \hat{k}_1^2}{\hat{p}_0^2} - 4 \frac{(k_0 \cdot \hat{k}_0 + k_1 \cdot \hat{k}_1)(p_0 + \hat{p}_0) + (k_0 \cdot \hat{k}_0 - k_1 \cdot \hat{k}_1)(p_1 - \hat{p}_1)}{(p_0 + \hat{p}_0)^2 + (p_1 - \hat{p}_1)^2} \]  

(3.45)

At times, due to the difference in orders \((q + 1 \text{ vs. } q)\), it may be necessary to compare a complex pole pair function with a single real pole function. From Appendix B, this integral is also shown to result in the real function of the poles and residues:

\[ E = \frac{\hat{p}_0(k_0^2 - k_1^2) + 2k_0k_1\hat{p}_1}{\hat{p}_0^2 + \hat{p}_1^2} + \frac{k_0^2 + \hat{k}_0^2}{2p_0} + \frac{\hat{k}_0^2 + \hat{k}_1^2}{\hat{p}_0} - 4 \frac{k_0 \cdot \hat{k}_0(p_0 + \hat{p}_0) + k_0 \cdot k_1 \hat{p}_1}{(p_0 + \hat{p}_0)^2 + \hat{p}_1^2} \]  

(3.46)
3.5 Frequency Scaling

In addition to the error associated with the AWE approximation, there is also the question of roundoff error. When the eigenvalues of $A^{-1}$ are very small, or very large, the powers of $A^{-1}$, and therefore the moments, change very rapidly. The large variation in moment values causes the moment matrix in equation (3.21) to become ill-conditioned and near singular. For such cases, higher orders of approximation can not be made unless the moment values are scaled.

As is done classically when working in the frequency domain[19] where values may range from $10^2$ to $10^8$ Hz or more, the frequency is scaled to first find a normalized solution, from which the normalized poles are scaled back to find the desired approximation. In AWE the normalization is chosen about the first pole by selecting a scale factor of:

$$\gamma = \frac{m_{-1}}{m_0}.$$  \hspace{1cm} (3.47)

This normalization is accomplished by replacing $s$ by $\gamma \hat{s}$ in the series

$$[X_h(s)]_i = \left[ -A^{-1}(As^{-1} + I + A^{-1}s + A^{-2}s^2 + \ldots)X_h(0) \right]_i,$$  \hspace{1cm} (3.48)

and approximating the response as before. The initial conditions are scaled by $\gamma^{-1}$ and the $j^{th}$ moment, $[m_j]_i$, is scaled by $\gamma^j$.

In practice the first order AWE approximation is made from $m_{-1}$ and $m_0$. If higher orders of approximation are sought, the $i^{th}$ state equations are scaled by the factor $\gamma_i$ from equation (3.47). The higher order moments are scaled by solving for succeeding moments from equation (3.31) using a scaled $A^{-1}$. From equation (3.29), $A^{-1}$ can be scaled by $\gamma$ most efficiently by scaling all of the energy storage element values. With all the energy storage elements scaled by $\gamma$, the scaled matrix, $\tilde{A}^{-1}$, is used as before to find the higher order scaled moments, $\tilde{m}_j$. From equations (3.21) and (3.17) the normalized poles and residues are found. The actual approximating poles and residues are then determined by $p = \gamma \tilde{p}$ and $k = \gamma \tilde{k}$. 
Chapter 4

Relation to RC Tree Methods

To demonstrate that AWE is not an excessively complex process, it is applied to the linear RC tree delay estimation problem from which it evolved. It will be shown that in general, a first order AWE approximation for an RC tree yields the Elmore delay as the reciprocal dominant pole.

4.1 First Order AWE Approximation

Finding $A^{-1}$ from the state equations (3.1) is equivalent to solving for the port voltages of the open circuit capacitance ports and the short circuit inductance ports[18]. For many circuit configurations, RC trees included, solving for these port variables is trivial.

Consider as an example the RC tree shown in Figure 4.1. The state equations for this RC circuit can be expressed in matrix form as

$$\dot{v} = C^{-1}Gv + C^{-1}Bu(t),$$

(4.1)

where $C$ is the diagonal capacitance matrix and $G$ is the related port conductance matrix. The steady state and the moments for this circuit can be obtained from the circuit in Figure 4.2, where all the capacitors in Figure 4.1 have been replaced
by current sources. The steady state, which corresponds to the $m_{-1}$ term since the

![RC Tree Circuit](image)

Figure 4.1: RC tree example circuit.

initial conditions are zero, is obtained by setting $\dot{v}$ equal to zero in equation (4.1) and solving for the capacitor voltages. This solution is equivalent to opening all the current sources $I$ in Figure 4.2 and obtaining the voltages across them.

![Capacitor Ports](image)

Figure 4.2: Capacitor ports are replaced by current sources.

The homogeneous solution to (4.1) is (from equation (3.7)):

$$v_h(s) = -G^{-1}C(I + G^{-1}Cs + G^{-2}Cs^2 + \ldots)(-v_{ss}(0))$$

(4.2)

The $m_0$ moment is obtained by setting $u(t)$ equal to zero and $\dot{v}$ equal to $Cv_{ss}(0)$ and solving for $v$ from equation (4.1). This solution is equivalent to setting $I$ in Figure
4.2 equal to \(-C v_{ss}(0)\) and \(u(t)\) equal to zero, then obtaining the node voltages. Only \(m_{-1}\) and \(m_0\) are needed for a first order approximation, but succeeding moments could be found by similar recursion if higher orders of approximation were sought. The next moment, \(m_1\), can be obtained by setting \(I\) equal to \(C m_0\) with \(u(t) = 0\) and solving for the node voltages, and so on. Thus, finding the moments of the actual circuit in Figure 4.1 is a succession of dc solutions to the circuit in Figure 4.2. The equations describing this dc circuit must be formulated and solved only once to determine the steady state. Solution for the moments then requires only changing the dc inputs for the new dc solution.

Since solving for the circuit moments requires only successive dc analyses, in practice the state equations are not formulated. Moreover, for simple circuits such as RC trees, the steady state solution is explicit and the first moment, or Elmore delay can be determined by a tree walk of the circuit graph[7]. A graph representing the circuit in Figure 4.2 is shown in Figure 4.3. The voltage sources and the resistors form a spanning tree, i.e. a graph that touches all nodes but forms no cycles. In [7] it was shown that calculation of the first moment for any node is \(O(n)\), where \(n\) is the number of capacitors. The Elmore delay at \(C_4\) as calculated from the graph is:

\[
T_D^4 = (R_1 + R_3 + R_4)C_4 + (R_1 + R_3)C_3 + R_1C_2 + R_1C_1
\]  

(4.3)

A tree walk is viable for RC trees, but does not provide for a general analysis of
paths with floating capacitance or inductance. With AWE, tree link analysis[27] is employed to solve for the moments since it enables a solution for all circuit topologies. It will be shown for the case of an RC tree, however, that tree link analysis provides for a generalized tree walk. For the RC tree in Figure 4.2, the spanning tree in Figure 4.3 is equivalent to the fundamental tree which uniquely specifies the tree link equations. From these equations the circuit moments can also be obtained in linear time. The tree link graph in Figure 4.3 has the following fundamental loop/cutset matrix $F[27]:$

$$
\begin{bmatrix}
-1 & -1 & -1 & -1 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1
\end{bmatrix}
$$

(4.4)

The overall solution for the circuit of Figure 4.2 can be obtained easily in terms of either the tree branch voltages or the link currents. For an RC tree all of the links (open capacitances) are current sources and therefore the solution for the link currents is trivial,

$$i_l = I.$$

(4.5)

The state variables, or link voltages, are then obtained from

$$v_l = -FTRF + FTV_s,$$

(4.6)

where $R$ is a diagonal matrix of the tree branch resistances, $V_s$ is a diagonal matrix of tree branch voltage sources and $I$ is the vector of link currents for this RC tree from equation (4.5). The matrix $FTRF$ can be formed by inspection of the tree link graph, or $F$ matrix, as described in [28,29]. Equation (4.6) for this circuit is

$$v_l = \begin{bmatrix}
-R_1 & -R_1 & -R_1 & -R_1 \\
-R_1 & -(R_1 + R_2) & -R_1 & -R_1 \\
-R_1 & -R_1 & -(R_1 + R_3) & -(R_1 + R_3) \\
-R_1 & -R_1 & -(R_1 + R_3) & -(R_1 + R_3 + R_4)
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix}
- \begin{bmatrix}
u(t) \\
u(t) \\
u(t) \\
u(t)
\end{bmatrix}
$$

(4.7)
4.1. **FIRST ORDER AWE APPROXIMATION**

The $m_{-1}$ moment, or steady state, can be found from (4.7) by setting $I$ equal to $0$ and solving for the resulting explicit expression for $v_l$. If $u(t)$ is a five volt step input,

$$v_l = v_{ss} = -m_{-1}$$

(4.8)

in this case a vector all entries of which are five. The $m_0$ moment is obtained by setting $I$ in equation (4.6) equal to $Cv_{ss}$ and $V_s$ equal to $0$ and solving for $v_l$ for all the nodes of interest. Solving $v_l$ for all four state variables results in:

$$v_l = -\begin{bmatrix}
-R_1(C_1 + C_2 + C_3 + C_4) \\
-R_1(C_1 + C_2 + C_3 + C_4) - R_2 C_2 \\
-R_1(C_1 + C_2 + C_3 + C_4) - R_3(C_3 + C_4) \\
-R_1(C_1 + C_2 + C_3 + C_4) - R_3(C_3 + C_4) - R_4 C_4
\end{bmatrix} = \begin{bmatrix}
T_D^1 \\
T_D^2 \\
T_D^3 \\
T_D^4
\end{bmatrix}$$

(4.9)

which are the Elmore delays for the respective nodes. From equations 4.7 and 4.9 it is apparent that finding the Elmore delays via tree link analysis is also $O(n)$, as is the case for a tree walk.

Once the first two moments $m_{-1} (V_{ss})$ and $m_0 (T_D)$ are determined, a first order approximation can be made for the node voltages. The homogeneous response at node four is approximated by the first order model:

$$\dot{V}_4(s) = \frac{k_1}{(s - p_1)} = \frac{-k_1 \tau_1}{(1 - s \tau_1)} = -k_1 \tau_1 (1 + s \tau_1 + s^2 \tau_1^2 + \ldots).$$

(4.10)

Equating the first two moments of this model to that of the actual circuit

$$k_1 = [m_{-1}]_4 = -5.$$  

(4.11)

and

$$-k_1 \tau_1 = [m_{-1}]_4 = R_1(C_1 + C_2 + C_3 + C_4) + R_3(C_3 + C_4) + R_4 C_4.$$  

(4.12)

The first order AWE step response approximation for the voltage at $C_4$ is:

$$v_4 = v_{p4} + v_{h4} = 5 - 5e^{-t/\tau_1},$$

(4.13)
where $\tau_1$ is equal to the Elmore delay. Equation (4.13) is compared with the SPICE response for this circuit in Figure 4.4.

We have shown that a first order AWE analysis is equivalent to those RC tree methods that utilize Elmore's delay expression. In addition, solving for the $m_0$ term at $C_4$, or $T_D^4$, by way of tree link analysis was shown to be equivalent to a tree walk as described in [7]. More importantly though, when the path is such that a walk is not possible, e.g. it may contain a floating capacitor, it will be shown that tree link analysis continues to apply without loss of generality.

### 4.2 Inexplicit Steady State Solution

An explicit solution to the circuit with capacitors replaced by current sources and inductors replaced by voltage sources is also possible for circuit configurations other than RC trees. Any RLC circuit for which the tree can be specified by only inductors, or the links can be specified by only capacitors has a trivial solution. For
instance, the "RLC Ladder" shown in Figure 4.5 can be solved explicitly since all of the links are capacitors.

![RLC Ladder Diagram](image)

Figure 4.5: RLC Tree which has a trivial steady state solution.

There are cases, such as a resistor to ground with the RC tree in Figure 4.6, which actually require finding the LU factors since the steady state is no longer explicit and the links are not exclusively capacitors. Irrespective of the method used to approximate the transient response waveform, the steady state must be determined \textit{a priori}. Tree link analysis recognizes when the steady state solution is

![RC Circuit Example](image)

Figure 4.6: RC circuit example with grounded resistor.

not explicit and formulates the problem to solve for the least number of variables. With the capacitors replaced by current sources as shown in Figure 4.8, the tree link graph for this circuit is now as shown in Figure 4.7. The resistors form a cycle in the graph, hence one of them, for this example $R_5$ (conductance $G_5$), must be entered as a link.
Figure 4.7: Tree Link graph when the dc solution is not explicit.

The link currents can be found from the loop equations

\[ I_i = -G F^T R F_i + I_i \]  \hspace{1cm} (4.14)

where \( G \) is the diagonal conductance matrix:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_5 \\
\end{bmatrix}
\]  \hspace{1cm} (4.15)

The first four link current expressions are again explicit, and the expression for \( I_{G_5} \) can be obtained with a complexity of \( \mathcal{O}(n) \). Thus calculation of the steady state and first moment with the inclusion of a grounded resistor is still linear in circuit size.

In Chapter 2, the extension of RC tree methods to include the effects of grounded resistance[10,15,16] were described. Essentially the Elmore delay, or first moment, was scaled by the steady state voltage as described by equation 2.9. From equation 4.2 it is apparent that the first moment is changed not only by the change in steady state, \( v_{ss}(0) \), as reflected by the change in \( x_h(0) \), but also by the change in \( G^{-1} \). With \( R_5 = 4\Omega \), the first order AWE approximation is compared with the SPICE response in Figure 4.9.
4.2. INEXPlicit STEADY STATE SOLUTION

Figure 4.8: Capacitors replaced by current sources.

Figure 4.9: Response for RC tree with grounded resistor.
4.3 Finite Input Rise Time

Finite input signal voltage rise times can have a significant, even dominant impact on the overall response waveform. RC tree methods typically apply only to step response approximations. Finite input voltage slope effects have been considered by adding the input signal rise time to the Elmore delay to approximate the overall delay[30]. A more generalized approach for including input rise time effects is available with AWE.

Consider the RC tree in Figure 4.1 driven by a five volt input signal with a rise time of 1ms. Because the RC tree is linear, AWE can approximate this circuit response by superposing the results from positive- and negative-going ramp inputs as shown in Figure 4.10. Only the positive ramp solution needs to be found since the negative ramp response has the same solution but is of opposite sign and shifted in time by 1ms.

The particular solution at node 4 for the positive-going ramp is

\[ v_p(t) = 5 \times 10^3 t - 3.5 \times 10^4. \]  \hspace{1cm} (4.16)

The first order AWE approximation for the homogeneous solution at node 4 is

\[ v_h(t) = 3.5e^{-1.667t}. \]  \hspace{1cm} (4.17)

The complete response approximation is the combined response from equations (4.16) and (4.17) for the positive and negative ramps.
Equations (4.18) and (4.19) are shown plotted in comparison to the SPICE response in Figure 4.11. The first order AWE ramp response approximation makes an excellent prediction of the delay. The largest error in this waveform approximation occurs near time \( t=0 \), as shown by the plot in Figure 4.12. This error is to be expected since the AWE approximation is matching the frequency expansion about \( s = 0 \) \( (t = \infty) \). From Figure 4.12 it is apparent that the AWE approximation starts out with a negative slope. In reality, this is not possible for an RC tree response when there are equilibrium initial conditions. The problem is that the initial boundary conditions for the case of a ramp input have not been met completely. For the case of a step response approximation the initial boundary conditions for the current as well as the voltage are met by matching the \( m_{-1} \) term. To ensure that the same is
true for a ramp input both the $m_{-1}$ and the $m_{-2}$ terms must be matched. Matching the $m_{-2}$ term is tantamount to ensuring that the first derivative of the approximate voltage response matches the first derivative of the actual voltage response at time $t=0$. This extended matching guarantees that the initial slope at time $t=0$ is of the correct sign. For most timing analysis applications the possible error in voltage slope at time $t=0$ does not affect the delay estimate. But if necessary, this glitch can be removed by proper matching of the $m_{-2}$ terms. However, as more positive moments are matched, i.e. the order of approximation is increased, the initial slope at $t = 0$ better approaches exact. This phenomenon will be demonstrated by several ramp response examples in the next chapter. Moreover, the relationship between the circuit time constants and the input signal rise time will be shown in section 5.2 for an RC tree example circuit.
4.4 Increased Orders of Approximation

The first order step response approximation in Figure 4.4 exhibits an error which may be unacceptable for some delay applications. In [7], what corresponds to a first order AWE response waveform is bounded to what are sometimes very pessimistic max/min values. Since with AWE higher orders of approximation can be found at an incremental cost to the first order approximation, the order of approximation is increased until an acceptable error term exists. For the first order approximation, the error term as described in section 3.4, is calculated to be 36%. A second order approximation for the RC tree in Figure 4.1 can be obtained upon calculating the next two moments. The second order AWE unit step response approximation is compared with the SPICE response in Figure 4.13. The error term is decreased to 1.6%. The AWE and SPICE response plots are indistinguishable at the resolution shown. Higher orders of approximation are obviously desirable for improving the

![Figure 4.13: Second order step response approximation.](image-url)

accuracy of the response approximation. More importantly, though, higher orders of approximation are necessary for the general handling of nonmonotone response
waveforms arising from circuits which contain multiple input signals, nonequilibrium initial conditions, floating capacitors, complex poles, etc.. In the chapter which follows several examples are used to demonstrate AWE's ability to analyze these types of circuit responses.
Chapter 5

General RLC Circuit Examples

A first order AWE approximation has been shown to be equivalent to some of the RC tree methods. The benefit of AWE over RC tree methods is the ability to recognize and handle more complex interconnect models without loss of generality. In this chapter some linear RLC circuits are used to demonstrate the applicability of AWE for solving general linear interconnect models.

5.1 MOS Interconnect

Low- to mid-frequency MOS circuit interconnect can be modeled well with an RC tree. The RC tree in Figure 5.1 is a typical example of such a model. Of particular interest is the widely varying time constants for this circuit. "Stiff" circuits such as this RC tree are normally troublesome for circuit[8] and timing[31,32] simulators; however, in AWE the small time constants are not obtained if they are not required for the delay estimation. For the case of no initial nonzero voltages and a positive input with a slope of 1ns, the first order AWE approximation for the voltage across $C_T$ is shown and compared with that of SPICE in Figure 5.2. The error term (from section 3.4) is calculated to be 4.4%. A second order approximation is made by determining the next two moments. The second order approximation is shown in
CHAPTER 5. GENERAL RLC CIRCUIT EXAMPLES

Figure 5.1: RC tree with widely varying time constants.

Figure 5.2: First order approximation for capacitor $C_7$'s voltage.
Figure 5.3. At the resolution shown, this approximation and the SPICE response are difficult to distinguish. The error term is decreased to 0.15%.

![Graph](image)

Figure 5.3: Second order approximation for capacitor $C_7$'s voltage.

Higher orders of approximation not only provide an improved waveform estimate but also enable a measure of the accuracy of the first order approximation. This capability is essential for interconnect models in general, since there may be complex poles or low frequency zeros which render a first order approximation useless. Moreover, as described in Chapters 3 and 4, the higher orders of approximation are obtained at an incremental cost to the first order estimate. For example, the cost of a second order approximation as compared to the first order estimate for this circuit is shown in Figure 5.4. The first order approximation time is the CPU time required to set up the equations, find the steady state and $m_0$, and solve for the dominant pole and residue. The second order approximation incremental CPU time is that required to find the next two moments, and the two approximating poles and residues.

The approximate poles for the first and second order approximations are given
Figure 5.4: CPU time comparison between 1st and 2nd order approximation along with the actual poles in Table 5.1. The first order AWE analysis approximates the dominant pole at a value very close to the actual dominant pole. The second order approximation finds two poles very close to the first two actual poles. In general, as the order of the approximation is increased, the approximate poles are found to "creep up on" the actual circuit poles as demonstrated by this example.

5.2 Finite Input Signal Rise Time

The effect of the input voltage rise-time on the delay is not always negligible. In fact, the relative difference between the rise-time and the time-constants of the circuit determines which more dominates the resulting delay. As the input voltage rise-time is decreased, the input slope becomes much larger than the reciprocal of the circuit time constants; i.e., the time constants dominate the delay and the response quickly approaches the unit step response. With an input rise-time of .1ns for the circuit in Figure 5.1, the second order AWE approximation is compared with a SPICE simulation in Figure 5.5. When the input slope becomes smaller, however, the circuit will eventually become a ramp follower and the delay is primarily a function of the input slope. For a voltage input with a rise-time of 10ns applied to the circuit in Figure 5.1 the second order AWE approximation is shown in Figure 5.6. For this case, and in general for circuits with real poles, when the input slope
### Table 5.1: Approximating and exact poles for RC tree example.

<table>
<thead>
<tr>
<th>no initial conditions</th>
<th>$V_{ce}(t=0)=5.0$ v</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st order</td>
<td>2nd order</td>
</tr>
<tr>
<td>-1.7358e9</td>
<td>-1.7818e9</td>
</tr>
<tr>
<td>-1.2572e10</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
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<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5.5:** Input voltage rise-time of .1ns.
becomes smaller than the poles of the circuit, the delay becomes dominated by the input voltage rise time.

5.3 Nonequilibrium Initial Conditions

Chu and Horowitz[12] proposed a two time constant model for a class of RC trees with nonequilibrium initial conditions. With AWE arbitrary nonequilibrium initial conditions and charge sharing are handled for general RLC circuits.

The initial state of the circuit may cause charge to be shared between capacitors which can affect the delay at various nodes. It is well known that the initial state of a circuit, \( x_0 \), can excite or suppress various of its natural frequencies[27]. With AWE the moments are functions of the initial conditions \( x_0 \) so as to include this effect. The dominant pole approximations are therefore determined by the initial state as well as the circuit elements. With the initial voltage of \( C_6 \) in Figure 5.1 equal to five volts, the first and second order approximate poles that result are shown in Figure 5.6: Input voltage rise-time of 10ns.
5.3. **NONEQUILIBRIUM INITIAL CONDITIONS**

Table 5.1. The AWE approximation for the case of $v_6(t = 0) = 0$ found the two most dominant poles to be very near the first two actual poles. With $v_6(t = 0) = 5$, however, the initial conditions introduce a low-frequency zero which partially cancels the second pole. The AWE approximation finds the two most dominant poles to be near the first and third actual poles when $v_6(t = 0) = 5$. The first and second order approximate waveforms determined by these approximate poles are shown in Figures 5.7 and 5.8 respectively. Obviously, a first order approximation, or single exponential function, cannot be used to approximate this non-monotone response. The error term for this first order approximation is 150%. The second order AWE response, which has an error estimate of 0.65%, is indistinguishable from the SPICE response at the resolution of this plot.

![Figure 5.7: First order approximation with $v_{e6}(0) = 5.0$.](image-url)
Figure 5.8: Second order approximation with $v_{ref}(0) = 5.0$.

5.4 Floating Capacitors

Although floating capacitors do not usually appear in digital signal paths directly, the charge that may be dumped to other paths due to coupling capacitance cannot always be neglected. In MOS technologies the floating capacitors which model the gate-drain can sometimes significantly affect the delay. For example, consider the RC tree circuit in Figure 4.1 with a floating capacitor connected to the output node as shown in Figure 5.9. The second order approximation for the voltage at $C_7$ is shown in Figure 5.10. The delay, taken to be the point at which a logic threshold of 4.0 volts is reached, changes from 1.6ns to 1.7ns because of charge sharing through $C_{11}$ to $C_{12}$. Notice too, that this second order approximation is not as good as the response approximation in Figure 5.3. This is reflected in the error term which is now 15% with the floating capacitance path, as compared to 0.15% without it. From second to third order the error term reduces from 15% to 0.14%.

The charge dumped onto $C_{12}$ is shown in Figure 5.11. Note that since we match the $m_0$ term of the actual response, the area under these voltage curves, hence the
Figure 5.9: RC tree with floating capacitor.

Figure 5.10: Second order approximation for capacitor $C_7$'s voltage.
charge transferred, is always exact.

Figure 5.11: Second order approximation for capacitor $C_{12}$’s voltage.

5.5 Inductors

For an example of a circuit with complex poles consider the analysis of the underdamped RLC circuit in Figure 5.12. This circuit is characterized by three pairs of complex poles as shown in Table 5.2. A first order AWE approximation for a circuit with dominant complex poles produces uninteresting results. The non-monotone homogeneous response cannot be modeled by a single exponential function. A second order approximation is required minimally.

The input voltage is a 5 volt ideal step. A first order approximation produces a single real dominant pole at $p = -2.833e9$. The error term for this first order approximation is large, 74%. A second order AWE analysis yields the approximating poles shown in Table 5.2. This dominant complex pole pair is near the actual first pole pair shown in Table 5.2. The second order AWE approximation is compared
5.5. **INDUCTORS**

![RLC Circuit Diagram](image)

**Figure 5.12:** RLC underdamped circuit with complex dominant poles.

<table>
<thead>
<tr>
<th>2nd order</th>
<th>4th order</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.0881e9</td>
<td>-2.6125e9j</td>
<td>-1.3532e9</td>
</tr>
<tr>
<td>-1.0881e9</td>
<td>+2.6125e9j</td>
<td>-1.3532e9</td>
</tr>
<tr>
<td>-7.3532e8</td>
<td>-6.7541e9j</td>
<td>-8.194e8</td>
</tr>
<tr>
<td>-7.3532e8</td>
<td>+6.7541e9j</td>
<td>-8.194e8</td>
</tr>
<tr>
<td></td>
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<td>-3.278e8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-3.278e8</td>
</tr>
</tbody>
</table>

**Table 5.2:** RLC circuit poles and approximate poles.
with the SPICE response in Figure 5.13. At second order AWE is able to detect

the overshoot but there is still a significant waveform difference as compared to
the SPICE response. The error term at second order is 22%. It is only at fourth
order, with the approximating poles shown in Table 5.2, that the error term be-
comes less than 1% and all of the response waveform detail is matched. The fourth
order AWE response is also shown plotted in Figure 5.13. For the most part, this
approximation is coincident with the SPICE waveform plot.

The step response at $C_3$ was shown to be dominated by two pairs of complex
poles. The residues were such that both pairs of poles made significant contributions
to the response waveform. If the input voltage rise time is changed from 0 to 1nS, the
residues would be changed such that there is only one complex pole pair dominating
the response. The second order RLC circuit response to a 5 volt input with a 1nS
rise time is compared to the corresponding SPICE response in Figure 5.14. As in the
RC tree case, the rise time of the input signal affects the error of the approximation.
In general, the step response approximation will exhibit the largest error term since
its transient response is larger than for the case of finite input signal slope.

![Graph](image)

Figure 5.14: **AWE** 2nd order approximation with a 1nS input voltage rise time.

Underdamped RLC circuits do not always have dominant complex poles. There are cases when approximations above second order are necessary to detect any undershoot or overshoot of the response. For example consider the response of the underdamped RLC circuit of Figure 5.15. The input voltage $V_{in}$ is a 5 volt ideal step. The actual poles for this circuit are shown in Table 5.3. Since this RLC circuit is only slightly underdamped there are two real poles closer to the $j\omega$ axis than the complex conjugate pair. With such a pole configuration the voltage overshoot at capacitor $C_1$ is not observed until the order of the **AWE** approximation is 4 or greater. This effect results from the manner in which the method of moment matching selects the approximate poles. The pole approximations as they move out the real axis are shown in Table 5.3 for second, third and fourth order approximations. The third order **AWE** approximation is compared with SPICE in Figure 5.16. This approximation generates an error term of 87%. The fourth order approximation is compared with SPICE in Figure 5.17. It is only at fourth order that the overshoot
is detected.

Figure 5.15: RLC underdamped circuit with real and complex poles.

<table>
<thead>
<tr>
<th>2nd order</th>
<th>3rd order</th>
<th>4th order</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.2360e-3</td>
<td>-8.9139e-3</td>
<td>-8.9146e-3</td>
<td>-8.9146e-3</td>
</tr>
<tr>
<td>-5.8872e-1</td>
<td>-8.7260e-2</td>
<td>-1.0295e-1</td>
<td>-1.0295e-1</td>
</tr>
<tr>
<td>-8.1036e-1</td>
<td>-5.5566e-1</td>
<td>+8.9642e-1j</td>
<td>-5.5561e-1</td>
</tr>
<tr>
<td>-5.5566e-1</td>
<td>-8.9642e-1j</td>
<td>-5.5561e-1</td>
<td>-8.9648e-1j</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-9.7969</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-9.9980e1</td>
</tr>
</tbody>
</table>

Table 5.3: RLC circuit poles and approximate poles for response at $C_1$.

These RLC circuit examples were selected to demonstrate what can happen when the circuit has arbitrary complex pole configurations. These examples are in no way typical. For the circuit in Figure 5.15 it is more likely that the desired output response would be at $C_3$, not $C_1$. As with RC tree methods the error in approximation is always largest closest to the driven end of the line. The response at $C_3$ can be approximated well with only a first or second order AWE approximation. The AWE first and second order responses are compared with that of SPICE in Figure 5.18. The second order approximation and the SPICE response are indistinguishable from the plots shown.
Figure 5.16: AWE third order approximation of step response at $C_1$.

Figure 5.17: AWE fourth order approximation of step response at $C_1$. 
Figure 5.18: AWE 1st and 2nd order approximation of the response at $C_3$.

5.6 Ram Cell Example

As a final example consider the asymptotic waveform evaluation for the RAM cell in Figure 5.19. The stage shown in this figure indicates the probable partition of this RAM cell for timing analysis[33]. A RAM cell provides an excellent example because its timing is critical and it demonstrates AWE’s handling of charge sharing, grounded resistors and loops of capacitors as shown in the linear approximate RAM cell model in Figure 5.20. The input signal to the word line is a low to high voltage with a 0.1nS rise time to READ the stored 0 onto the precharged high bitline. A first order approximation is not possible for this circuit because the charge sharing between $C_{bit}$ and $C_{store}$ introduces a voltage spike and non-monotone response waveforms which can not be characterized by a single exponential function. The AWE second order approximations for the voltages at $C_{bit}$ and $C_{store}$ are shown along with the SPICE responses in Figure 5.21. The SPICE and AWE waveforms are indistinguishable at the resolution shown. Note that the charge sharing at $t=0$ is well modeled with only a second order approximation.
Figure 5.19: Possible logic stage for a CMOS RAM.

Figure 5.20: Linear approximate RAM cell circuit.
Figure 5.21: AWE 2nd order approximation for $C_{bit}$ and $C_{store}$. 
Chapter 6

Other Applications

Up to this point, we have demonstrated AWE only for the delay evaluation of RLC interconnect circuits. From the general description in Chapter 3, it is apparent that AWE can be used to approximate the response of any system described by a set of linear first order ordinary differential equations. Moreover, the manner in which AWE performs dominant pole(s)/zero(s) analysis to approximate the response makes it a natural for variable order pole/zero analysis of analog circuits[34].

6.1 Analog Circuit Design Analysis

A common practice in the initial stages of analog circuit design is to analyze the circuit by hand using s-domain models. For quick results this calculation might be performed by using open-circuit or short-circuit dominant time constant analysis[35,36]. Unfortunately, such analyses will only yield the first pole and assume that the first pole is dominant. For some designs the location of higher-order poles relative to the first pole can significantly affect the bandwidth and phase margin values. Hence, the choice arises between performing a complete hand analysis of the circuit or using numerical pole-zero analysis. While finding an analytical expression for the transfer function shows both the pole-zero locations and how to change the circuit
to improve the design, it is both time consuming and error prone. The alternative
is to watch how poles and zeros move by trial and error using repeated computer
analysis runs with a numerical pole-zero analysis program. One popular numerical
technique available for finding the individual contributions of circuit elements to the
poles and zeros is a modification to the Souriau-Frame algorithm [18]. Present im-
plementations with this technique are practically limited to second order due to the
complexity involved in obtaining higher-order information[37]. More recently, ap-
proximations beyond second order have been made possible[38], but the complexity
at higher orders is still larger as compared with AWE.

We have shown that with AWE the dominant poles and zeros of a linear cir-
cuit can be approximated efficiently to any order including the case of complex or
repeated roots. Thus, AWE can provide the same information that hand analysis
would, but more efficiently and without the error-related penalties.

For a linear circuit with an impulse voltage excitation:

\[
\dot{x} = Ax + b\delta(t)
\]  

(6.1)

where \(x\) is the n-dimensional state vector and \(\delta(t)\) is the input excitation.

The solution of (6.1) is

\[
X(s) = (sI - A)^{-1}b.
\]  

(6.2)

and can be expanded in a Maclaurin series:

\[
X(s) = -A^{-1}(I + A^{-1}s + A^{-2}s^2 + \ldots)b.
\]  

(6.3)

For an AWE approximation of an impulse response there are no initial conditions
which need be matched. Thus the first \(2q\) moments are used to determine an
approximate q-pole transfer function model.

For example, consider the AWE frequency response approximation for the
common-emitter amplifier in Figure 6.1 which has the small-signal equivalent circuit
model shown in Figure 6.2.
Figure 6.1: Common-emitter amplifier stage.

Figure 6.2: Small-signal linear circuit model.
The first order approximate pole and residue are found trivially from equations (3.21), (3.22) and (3.17), resulting in

$$\hat{\mathbf{X}}_1(s) = \frac{9.8648}{s + 6.2464e6}.$$  \hspace{1cm} (6.4)

A second order approximation is made using the first four voltage moments:

$$\hat{\mathbf{X}}_2(s) = \frac{9.9682}{s + 6.3091e6} + \frac{-9.9734e-2}{s + 8.7420e8}. \hspace{1cm} (6.5)$$

The second residue in equation (6.5) is two orders of magnitude smaller than the first residue, and the first term in equation (6.5) is not much different than that in equation (6.4). Because the magnitude of the second residue is negligible in comparison to the first we would expect that the first order approximation is very close to the second order approximation. Typically in such cases, a higher order of approximation would not be sought, but for this example a third order analysis was undertaken yielding the exact response

$$\mathbf{X}(s) = \frac{9.9682}{s + 6.3091e6} + \frac{-9.6175e-2}{s + 8.4174e8} + \frac{-3.5601e-3}{s + 2.8622e10}. \hspace{1cm} (6.6)$$

It is apparent from (6.6) that the circuit in Figure 6.2 has a dominant pole at $\sigma = -6.3091e6$ and a first order approximation would be expected to predict this circuit response well. Equation (6.4) predicts a bandwidth of 1MHz and a phase margin of 95°, while equations (6.5) and (6.6) estimate the bandwidth and phase margin to be 1MHz and 90°.

The circuit in Figure 6.1 was used in [35] as an example for dominant pole approximation methods. The results from the dominant pole analysis in [35] are similar to equation (6.4), but in [35] there is no means of testing the reliability of the dominant pole assumption. Conversely, AWE provides a means of checking the validity of the first order model by finding higher order response approximations. In addition, AWE is applicable when the poles are not widely spread, or when the dominant poles are repeated or complex.
As a second example consider solving for the frequency response of the feedback amplifier in Figure 6.3. The impulse response voltage $V_{out}$ is approximated from the small signal equivalent circuit in Figure 6.4. With $R_F = 2k$ this circuit has five real poles as shown in Table 6.1. The bandwidth for this circuit is 5.7MHz and the phase margin is $17^\circ$.

![Two-stage amplifier stage with feedback.](image)

**Figure 6.3: Two-stage amplifier stage with feedback.**

![Small-signal model for the two-stage amplifier with feedback.](image)

**Figure 6.4: Small-signal model for the two-stage amplifier with feedback.**

A first order approximation is

$$
\hat{X}_1(s) = \frac{-1.6828e2}{s + 3.031e7}.
$$

(6.7)

The bandwidth and phase margin from equation (6.7) are 4.8MHz and $92^\circ$. These values are far from exact mainly because the single pole is not as dominant as it was in the previous example. A second order approximation for this response is
\[
\hat{X}_2(s) = \frac{-2.2840e2}{s + 3.843e7} + \frac{6.0118e1}{s + 1.5391e8},
\]

which predicts a bandwidth of 5.75MHz and a phase margin of 15°. These values are very close to the actual bandwidth and phase margin for this fifth order system. A third order approximation has three terms, the first two of which are very close to those in equation (6.8) while the third has a residue of 1.6e-8. Thus we would not expect much change from second to third order.

Higher orders of approximation are necessary not only when the first pole is not dominant, but also to recognize when there may be repeated or complex poles. Consider for example the same circuit in Figure 6.4 with \( R_F \) now equal to 500Ω. The dominant poles for this circuit are a complex conjugate pair as shown in Table 6.1. The circuit now has a bandwidth of 22MHz.

<table>
<thead>
<tr>
<th>( R_F = 2k )</th>
<th>( R_F = 500 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.843e7</td>
<td>-1.051e8</td>
</tr>
<tr>
<td>-1.545e8</td>
<td>-1.051e8 ± 9.572e7j</td>
</tr>
<tr>
<td>-4.606e9</td>
<td>-4.771e9</td>
</tr>
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<td>-7.481e9</td>
<td>-7.467e9</td>
</tr>
<tr>
<td>-1.495e15</td>
<td>-1.510e15</td>
</tr>
</tbody>
</table>

Table 6.1: Poles for the Darlington small-signal model.

A first order approximation produces a single real dominant pole estimate of \( \sigma = -9.18e7 \), and predicts a bandwidth of 14.5MHz. Obviously, when the dominant poles are complex, a second order estimate is required minimally. A second order approximation finds the approximate complex conjugate poles \( s = -1.051e8 ± 9.548e7j \), and estimates the bandwidth to be 21.8MHz.

These examples illustrate that without knowledge of the exact poles, higher orders of approximation are sometimes necessary to detect pole clusters or complex
roots. AWE can provide the methodology for an analog design tool as demonstrated by an example in [34].

6.2 Future Work

6.2.1 Large Timestep Explicit Integration

Previous experience with the development of a PWL circuit simulator, TALISMAN [29], demonstrated that the very thing that makes this PWL simulator efficient in the dc simulation environment turned out to be a severe drawback in its attempted extension to the time domain environment.

To interpolate or extrapolate a valid solution in the transient time domain efficiently to the next corner of a PWL characteristic curve requires the use of an explicit integration algorithm[18]. Forward Euler is the simplest and most common of the explicit integration algorithms, but can be shown to be inaccurate and unstable for large timesteps. Implicit integration algorithms[18], while stable and potentially accurate, preclude the possibility of efficient interpolation or extrapolation to the next corner of the PWL model characteristic curves.

Unlike most traditional simulation strategies[8,18], which analyze circuits in the time domain through many steps of numerical integration, AWE attempts to perform only one frequency domain analysis to characterize a linear circuit via a few dominant poles and their residues. With these poles and residues, the time domain waveforms can be rendered explicit in terms of related exponential time domain functions. With one AWE analysis, a linear circuit is fully characterized, enabling the skipping over of unnecessary analyses to jump directly to the next time point when a PWL model is to reach a corner. A combination of AWE and Forward Euler may provide an efficient integration framework for PWL circuit simulation.
6.2.2 Distributed Elements and Circuit Partitioning

Since it is our intention that AWE can be applied to delay evaluation problems for high speed interconnect, it is desirable that AWE can analyze distributed as well as lumped elements. A simple approach for including distributed elements would be to approximate them by lumped N-stage approximations and solve for the delay as before. If the number of lumped stages is large, however, this strategy may prove to be inefficient.

A more general approach would be to take the exact s-domain solution for the distributed element and expand it into an infinite series about \( s=0 \). Then as in the lumped element case, match as many moments as necessary to a low order approximating function. Unfortunately, this approach requires that the distributed elements can be partitioned from the remainder of the circuit. Such partitioning within the AWE environment appears to be a formidable task. In general we are asking that any portion of the system which can be described by a function \( H(s) \) can be replaced by a lower-order model \( \hat{H}(s) \). With such capability any block described by an s-domain function could be included in the AWE simulation environment. This would make AWE a powerful methodology for mixed circuit/system level analog simulation.
Chapter 7

Conclusions

AWE is an efficient approach to waveform estimation for linear RLC interconnect circuit models. Floating capacitors, inductors, linear controlled sources, finite input rise-time, and charge sharing are all easily addressed in terms of AWE at any level of detail by merely increasing the order of the approximation. Because of its generality AWE should be applicable both to bipolar circuitry and printed circuit board level interconnect as well as to MOS circuit and interconnect timing estimation. In addition, the applications of AWE beyond that of delay evaluation, particularly for analog design, appear to be promising.
Appendix A

The recursion of the product

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
a_0 & a_1 & a_2 & \ldots & a_{q-1}
\end{bmatrix}
\begin{bmatrix}
m_{-1} \\
m_0 \\
m_1 \\
m_{q-3} \\
m_{q-2}
\end{bmatrix}
= 
\begin{bmatrix}
m_q \\
m_{q-1} \\
m_{2q-3} \\
m_{2q-2}
\end{bmatrix}
\]  \tag{A.1}

is most easily explained by considering the multiplications of \( A_{c}^{-1} \) on \( m_i \) individually.

The first product is

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
a_0 & a_1 & a_2 & \ldots & a_{q-1}
\end{bmatrix}
\begin{bmatrix}
m_{-1} \\
m_0 \\
m_1 \\
m_{q-3} \\
m_{q-2}
\end{bmatrix}
= 
\begin{bmatrix}
m_0 \\
m_1 \\
m_{q-2} \\
m_{q-1}
\end{bmatrix}
\]  \tag{A.2}

where

\[
m_{q-1}' = a_0 m_{-1} + a_1 m_0 + a_2 m_1 + \ldots + a_{q-1} m_{q-2}.
\]  \tag{A.3}
The second product is

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\begin{bmatrix}
m_0 \\
m_1 \\
m_2 \\
\vdots \\
m_{q-2} \\
\end{bmatrix}
= 
\begin{bmatrix}
m_1 \\
m_2 \\
m_{q-1} \\
\vdots \\
m_q \\
\end{bmatrix}
\]  

(A.4)

where

\[
m'_q = a_0m_0 + a_1m_1 + a_2m_2 + \ldots + a_{q-1}m'_{q-1}.
\]  

(A.5)

The \(q\)th product can be shown to be

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
\end{bmatrix}
\begin{bmatrix}
m_{q-2} \\
m'_{q-1} \\
m'_{q-1} \\
\vdots \\
m'_{2q-4} \\
\end{bmatrix}
= 
\begin{bmatrix}
m'_{q-1} \\
m'_{q} \\
m'_{q} \\
\vdots \\
m'_{2q-3} \\
\end{bmatrix}
\]  

(A.6)

where

\[
m'_{2q-2} = a_0m_{q-2} + a_1m'_{q-1} + a_2m'_q + \ldots + a_{q-1}m'_{2q-3}.
\]  

(A.7)

Upon setting the right hand side of equation (A.6) equal to the right hand side of equation (A.1), equation (3.21) results.
\[
\begin{bmatrix}
m_{-1} & m_0 & \ldots & m_{q-2} \\
m_0 & m_1 & \ldots & m_{q-1} \\
\vdots & \vdots & \ddots & \vdots \\
m_{q-2} & m_{q-1} & \ldots & m_{2q-4} \\
m_{q-2} & m_{q-1} & \ldots & m_{2q-3}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{q-2} \\
a_{q-1}
\end{bmatrix}
= 
\begin{bmatrix}
m_{q-1} \\
m_q \\
\vdots \\
m_{2q-3} \\
m_{2q-2}
\end{bmatrix}
\]

(A.8)
Appendix B

\[ E = \int_0^\infty (ke^{pt} + k^*e^{pt} - \hat{k}e^{\hat{p}t} - \hat{k}^*e^{\hat{p}t})^2 dt \]  \hspace{1cm} (B.1)

The complex \( k \) and \( p \) terms for \( v(t) \) can be replaced in B.1 by:

\[ k = k_0 + k_1 j \]
\[ k^* = k_0 - k_1 j \]
\[ p = p_0 + p_1 j \]
\[ p^* = p_0 - p_1 j \]  \hspace{1cm} (B.2)

And similar notation can be used for the \( \hat{v} \) terms. Then taking the square of the four terms in B.1 and integrating the ten resulting products from 0 to \( \infty \) yields:

\[
E = \frac{p_0(k_0^2 - k_1^2) + 2k_0 k_1 p_1}{p_0^2 + p_1^2} + \frac{\hat{p}_0(\hat{k}_0^2 - \hat{k}_1^2) + 2\hat{k}_0 \hat{k}_1 \hat{p}_1}{\hat{p}_0^2 + \hat{p}_1^2} + \frac{k_0^2 + k_1^2}{p_0} - 4\frac{(k_0 \hat{k}_0 - k_1 \hat{k}_1 (p_0 + \hat{p}_0) + (k_0 \hat{k}_1 + k_1 \hat{k}_0 (p_1 + \hat{p}_1))}{(p_0 + \hat{p}_0)^2 + (p_1 + \hat{p}_1)^2}
+ \frac{\hat{k}_0^2 + \hat{k}_1^2}{\hat{p}_0} - 4\frac{(k_0 \hat{k}_0 + k_1 \hat{k}_1 (p_0 + \hat{p}_0) + (k_0 \hat{k}_1 - k_1 \hat{k}_0 (\hat{p}_1 - p_1))}{(p_0 + \hat{p}_0)^2 + (\hat{p}_1 - p_1)^2} \hspace{1cm} (B.3)
\]

When a complex pair is matched to a single real pole \( (p_0) \) function:

\[ E = \int_0^\infty (k_0 e^{p_0 t} - \hat{k} e^{\hat{p}_0 t} - k^* e^{p_1 t} - \hat{k}^* e^{\hat{p}_1 t})^2 dt \]  \hspace{1cm} (B.4)

Replacing the \( \hat{k} \), \( \hat{p} \), \( \hat{k}^* \) and \( \hat{p}^* \) terms as before B.4 evaluates to:

\[
E = \frac{\hat{p}_0(\hat{k}_0^2 - \hat{k}_1^2) + 2\hat{k}_0 \hat{k}_1 \hat{p}_1}{\hat{p}_0^2 + \hat{p}_1^2} + \frac{k_0^2}{2p_0} + \frac{k_1^2}{2p_1} + \frac{\hat{k}_0^2 + \hat{k}_1^2}{\hat{p}_0} - 4\frac{k_0 \hat{k}_0 (p_0 + \hat{p}_0) + k_1 \hat{k}_0 (p_1 + \hat{p}_1)}{(p_0 + \hat{p}_0)^2 + \hat{p}_1^2} \hspace{1cm} (B.5)
\]
Bibliography


