Second Order ODEs – Continuous Time

- More expressive than 1^{st} order:
  - 1^{st} order: RL and RC circuits (Decay, blowup)
  - 2^{nd} order: RLC circuits (Decay, blowup, oscilation)
- Many systems with oscilation behavior can be modeled by 2^{nd} order ODEs
  - ECE
  - MechE
  - population dynamics
  - :

General Form (IVP)

$$a_2 \frac{d^2 x(t)}{dt^2} + a_1 \frac{dx}{dt} + a_0 x(t) = f(t), \quad t \geq 0$$

$x(0) = x_0$

$\dot{x}(0) = \dot{x}_0$

Canonical Form (IVP)

$$\frac{d^2 x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t), \quad t \geq 0$$

$x(0) = x_0$

$\dot{x}(0) = \dot{x}_0$

- Existence of general solution
- Uniqueness of IVP

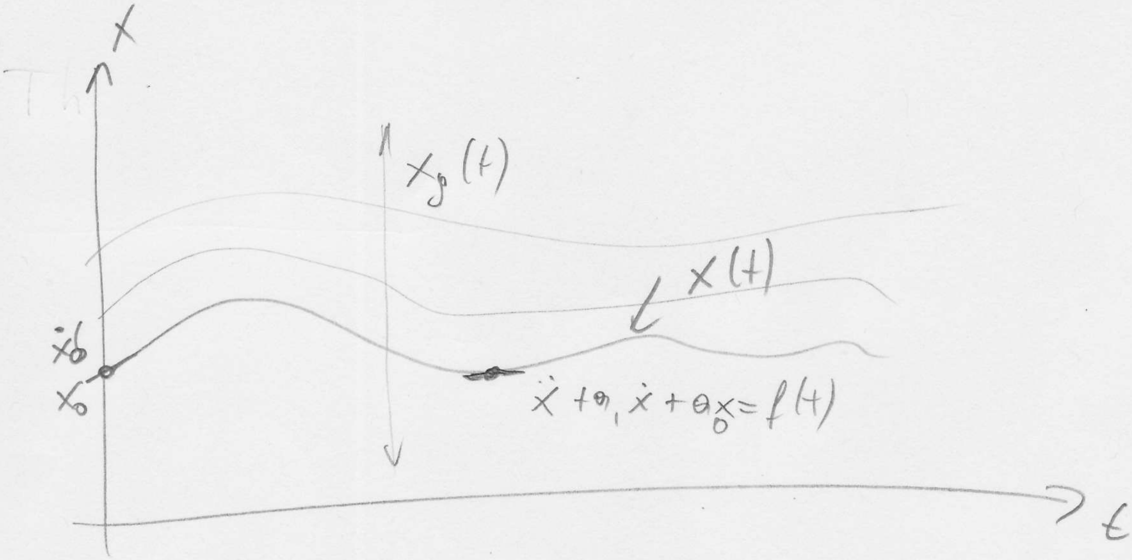
$\Rightarrow$ general solution has 2 parameter

$$\boxed{\dot{x} = \frac{dx(t)}{dt}, \quad \ddot{x} = \frac{d^2x(t)}{dt^2}}$$

Solution of ODE and IVP

Function $h(t)$ such that

- when applying LHS it is $\equiv$ RHS for $t \geq 0$] ODE
- At $t=0$ $h(t) = x_0$ and $\dot{h}(t) = \dot{x}_0$] IVP
- Solution to ODE without i.c. is general solution $x_g(t)$



Three properties

- Structure $x_g = x_h + x_p$
- Linearity w.r.t. initial condition
- Linearity w.r.t. forcing term

Structure of Solution

$x_g(t)$ general solution to $\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$

$\Rightarrow$ $\boxed{x_g(t) = x_h(t) + x_p(t)}$ with

- $x_h(t)$ is homogeneous solution:

  $x_h(t)$ general solution to $\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0$

  2 free parameters

- $x_p(t)$ is particular solution:

  one solution to $\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = f(t)$

  no parameter

— plug $x_g(t) + x_h(t)$ into ODE

to show that this holds

Linearity w.r.t. initial condition

Homogeneous IVP: with solution $x_h(t)$:

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 = 0 \quad , \quad t \geq 0, \quad x(0) = x_0 \;, \quad \dot{x}(0) = \dot{x}_0$$

$x_{h_1}(t)$ solution to IVP with $x(0) = x_{0_1}$, $\dot{x}(0) = \dot{x}_{0_1}$

$x_{h_2}(x)$ solution to IVP with $x(0) = x_{0_2}$, $\dot{x}(0) = \dot{x}_{0_2}$

if $\quad x_0 = \alpha_1 x_{0_1} + \alpha_2 x_{0_2}$

$\quad\;\; \dot{x}_0 = \alpha_1 \dot{x}_{0_1} + \alpha_2 \dot{x}_{0_2}$

Then

$$x_h(t) = \alpha_1 x_{h_1}(t) + \alpha_2 x_{h_2}(t)$$

Linearity w.r.t. intput

For ODE: particular solution $x_p(t)$

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 = f(t)$$

with

For ODE: particular solution $x_{p_1}(t)$

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 = f_1(t)$$

For ODE: particular solution $x_{p_2}(t)$

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 = f_2(t)$$

if

$$f(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t)$$

Then

$$x_p(t) = \alpha_1 x_{p_1}(t) + \alpha_2 x_{p_2}(t)$$

Four step solution

1) Homogeneous solution

$x_h(t) \longrightarrow$ characteristic polynomial

(ODE $\rightarrow$ algebraic problem)

root polynomial $\longrightarrow$ solve x_h

roots = eigenvalues, characteristic values, modes, natural frequencies

2) Particular solution

find a $x_p(t) \longrightarrow$ guessing method

variation of constants

separation of variables

3) General solution

$x_g(t) = x_h(t) + x_p(t)$

4) Impose i.c. (Solve IVP)

$$x_g(0) = x_0$$
$$\dot{x}_g(0) = \dot{x}_0$$
$\longrightarrow$ solve for the 2 parameter

Homogeneous ODEs and IVP

Setup

$$\ddot{x} + a_1 \dot{x} + a_0 = 0, \quad t \geq 0, \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

$$\Rightarrow \quad x_p(t) \equiv 0$$

Second Order ODEs have 2 characteristic freqs
- 2 real (distinct only in this course)
- 2 conjugate complex

Solution to characteristic polynomial

Example 1

$$\frac{d^2 x(t)}{dt^2} + 3 \frac{dx(t)}{dt} + 2x(t) = 0, \quad t \geq 0$$

$$x(0) = 0$$

$$\dot{x}(0) = 1$$

Step 1 Homogeneous Solution

Ansatz: $x_h(t) = \alpha e^{\lambda t}$

$$\Rightarrow \quad \dot{x}(t) = \alpha \lambda e^{\lambda t}$$

$$\ddot{x}(t) = \alpha \lambda^2 e^{\lambda t}$$

substitute into ODE:

$$\alpha e^{\lambda t}(\lambda^2 + 3\lambda + 2) = 0$$

- assume $\alpha \neq 0$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$ characteristic equation

Characteristic Polynomial

$$\Delta(\lambda) = \lambda^2 + 3\lambda + 2$$

obtain from ODE: $\frac{d^2}{dt^2} \longrightarrow \lambda^2$

$\frac{d}{dt} \longrightarrow \lambda$

Solution: solve quadratic equation

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2} = \begin{cases} -2 \\ -1 \end{cases}$$

$$\Rightarrow x_{h_1}(t) = \alpha_1 e^{-2t}$$

$$x_{h_2}(t) = \alpha_2 e^{-t}$$

both homogeneous solutions $\Rightarrow$ linearity gives general solution

$$x_h(t) = C_1 e^{-2t} + C_2 e^{-t} \quad t \geq 0$$

Step 2: particular solution

$$x_p(t) \equiv 0$$

Step 3: general solution

$$x_g(t) = x_h(t) = c_1 e^{-2t} + c_2 e^{-t}, \quad t \geq 0$$

Step 4: IVP

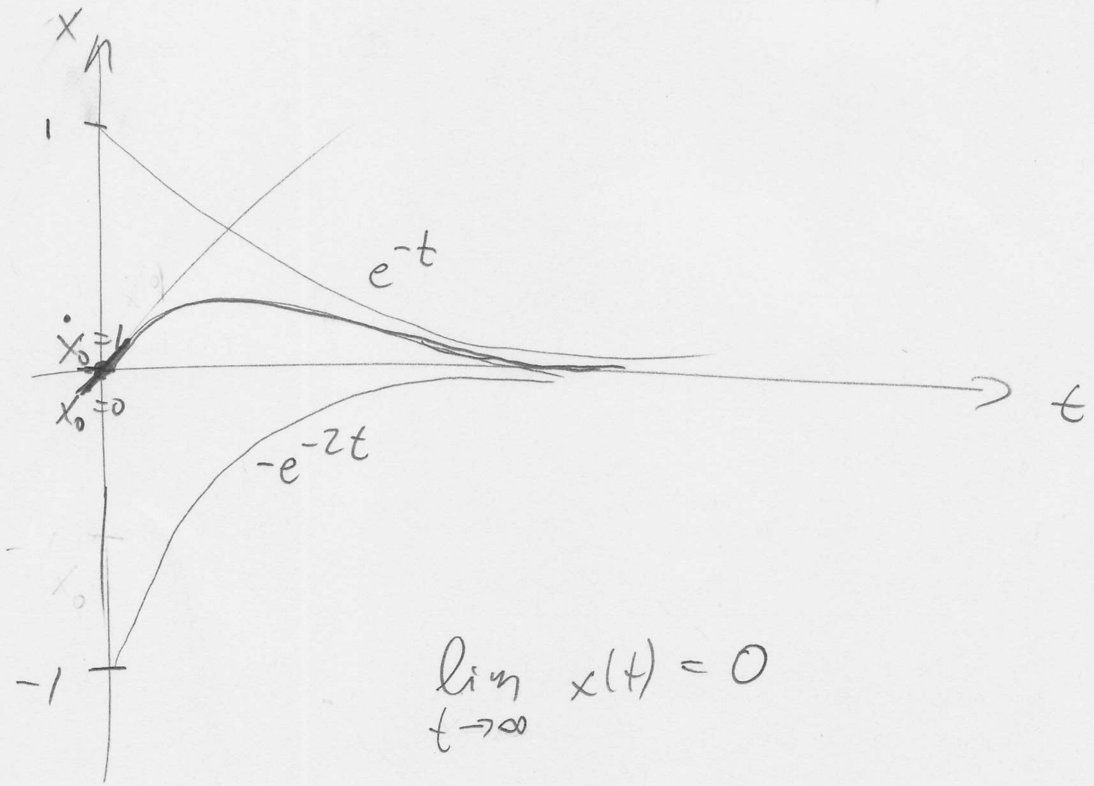
$$x(0) = c_1 + c_2 = 0$$

$$\dot{x}(0) = -2c_1 - c_2 = 1$$

$$\Rightarrow c_1 = -1$$

$$c_2 = 1$$

$$\Rightarrow \boxed{x(t) = -e^{-2t} + e^{-t}, \quad t \geq 0}$$



$$\lim_{t \to \infty} x(t) = 0$$

Example: Conjugate Complex natural frequencies

$$\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} + 2 = 0 \, , \quad t \geq 0 \, , \qquad \begin{array}{l} x(0) = 2 \\ \dot{x}(0) = -1 \end{array}$$

Step 1: Homogeneous Solution

$$\Delta(\lambda) = \lambda^2 + 2\lambda + 2$$

$$\Rightarrow \quad \lambda_{12} = -1 \pm j$$

$$x_{h_1}(t) = c_1 e^{-(1-j)t} \, , \qquad x_{h_2}(t) = c_2 e^{-(1+j)t}$$

$$x_h(t) = c_1 e^{-(1-j)t} + c_2 e^{-(1+j)t} \, , \qquad t \geq 0$$

Step 2: Particular solution : $x_p(t) \equiv 0$

Step 3: general solution : $x_g(t) = x_h(t)$

Step 4: IVP

$$x(0) = c_1 + c_2 = 2$$

$$\dot{x}(0) = (-1+j)c_1 + (-1-j)c_2 = -1$$

$$\Rightarrow \quad c_1 = 1 - \tfrac{1}{2}j$$

$$c_2 = 1 + \tfrac{1}{2}j$$

$$\boxed{x_h(t) = \left(1 - \tfrac{1}{2}j\right) e^{-(1-j)t} + \left(1 + \tfrac{1}{2}j\right) e^{-(1+j)t}}$$

We know $x_h(t)$ is real valued

Use conjugation identities:

$$c_1^* \cdot c_2^* = (c_1 c_2)^*$$

$$c_1 + c_1^* = 2\,\mathrm{Re}(c_1)$$

$$\Rightarrow \; \underline{x_h = 2\,\mathrm{Re}\left((1 - \tfrac{1}{2}j)\, e^{-(1-j)t}\right)}$$

Compute polar form of c_1:

$$c_1 = 1 - \tfrac{1}{2}j = |1 - \tfrac{1}{2}j|\, e^{j\theta} = \frac{\sqrt{5}}{2} e^{j \underbrace{\tan^{-1}(-\frac{1}{2})}_{-26.56^\circ}}$$

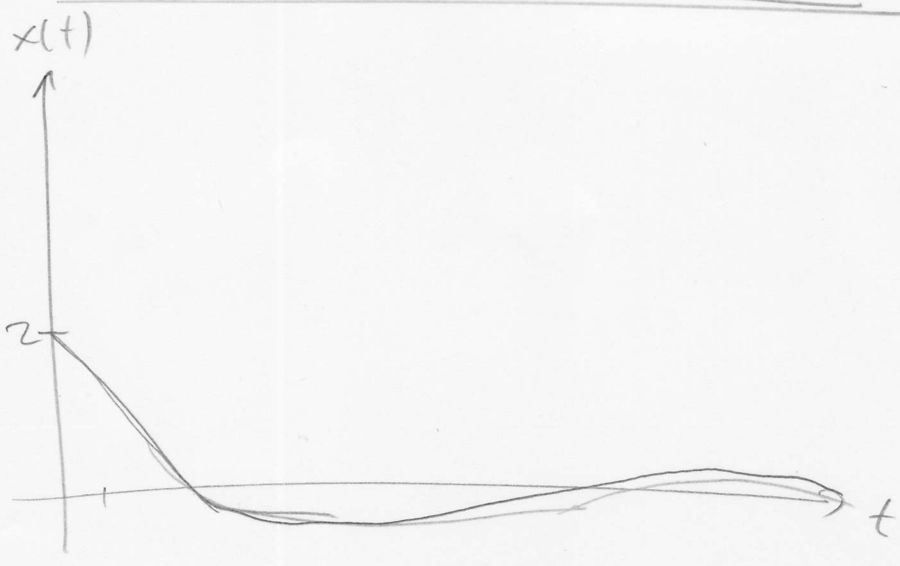
$$= \underline{\frac{\sqrt{5}}{2} e^{-j\,26.56^\circ}}$$

IVP solution:

$$x(t) = 2\,\mathrm{Re}\left(c_1 e^{-(1-j)t}\right)$$

$$= 2\,\mathrm{Re}\left(\frac{\sqrt{5}}{2} e^{-j26.56^\circ} e^{-t} e^{jt}\right) = \sqrt{5}\, e^{-t}\, \mathrm{Re}\left(e^{j(t-26.56^\circ)}\right) \overset{\text{Euler's Formula}}{=}$$

$$= \underline{\sqrt{5}\, e^{-t} \cos(t - 26.56^\circ)\,, \quad t \geq 0}$$



General Case / Summary

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = 0, \quad t \geq 0$$

$x(0) = x_0$

$\dot{x}(0) = \dot{x}_0$

Step 1: Homogeneous solution

characteristic Polynomial:

$$\Delta(\lambda) = \lambda^2 + a_1\lambda + a_0$$

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}, \quad a_1^2 - 4a_0 > 0$$

$$\Rightarrow x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Step 2: Particular solution : $x_p(t) \equiv 0$

Step 3: general solution : $x_g(t) = x_h(t)$

Step 4: IVP

$$c_1 + c_2 = x_0$$

$$\lambda_1 c_1 + \lambda_2 c_2 = \dot{x}_0$$

$\Rightarrow$ solve equation

$\Rightarrow$ find real representation (lots of algebra)

Closed form Solution of IVP

Use $\lambda_1 = \sigma_1 + j\omega_1$

$$x(t) = \underbrace{\frac{\sqrt{(\dot{x}_0 - \sigma_1 x_0)^2 + \omega_1^2 x_0^2}}{\omega_1}}_{\text{const}} e^{\sigma_1 t} \cos\left(\omega_1 t - \underbrace{\tan^{-1}\frac{\dot{x}_0 - \sigma_1 x_0}{\omega_1 x_0}}_{\text{const}}\right)$$

$$\boxed{= C_1 e^{\sigma_1 t} \cos(\omega_1 t - C_2)}$$

amplitude

exponential decay

frequency

phase shift