An Equivalent Circuit Formulation for Three-Phase Power Flow Analysis of Distribution Systems

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Abstract — **In this paper, we describe a power flow formulation for 3-phase distribution systems that is based on an equivalent circuit model. It is shown that this physical model based solution is able to accommodate a wide range of complex and unbalanced loads without loss of generality. The approach is an extension of the single phase formulation in [1] that uses current and voltage as the state variables. This formulation is shown to provide excellent modeling efficiency for distribution system components, such as induction motors that can be modeled as linear circuit elements. The formulation is further capable of incorporating complex nonlinear models to capture more details or represent future bus models. A challenging IEEE 4-bus test case is used as a proof of concept to demonstrate the efficacy of this approach.**

Index Terms—**three-phase power flow, smart grid, distribution systems**

I. Introduction

Single phase power flow methods based on iteratively solving the power mismatch equations were first conceived of decades ago [2], [3] and remain the standard for simulating transmission-level power grids, where perfect phase balance is assumed. At the distribution level, however, these methods experience poor convergence due to the radial distribution feeders and difficulties in handling unbalanced loads and certain load models, such as the constant impedance load model that appears as nonlinear in the power mismatch formulation [4]. The distribution problem can be instead formulated in terms of the current mismatch equations and using symmetrical component transformation techniques. Even though most of the current mismatch equations are linear, there are many cases [7] [8] where the current distribution system simulation tools either fail to converge or have difficulty handling specific configurations.

Most importantly, these distribution formulations are mathematically derived, not physical model based, which can sometimes fail to produce natural governing power system (circuit) equations. For example, analyzing a nonlinear open-Wye connected unbalanced PQ load with fixed-point iteration, which is the method proposed for many distribution problems [4][9], results in a "physical model" corresponding to three nonlinear independent current sources that are connected to a "floating node." This represents an "unnatural circuit" configuration that is recognized to have convergence problems.

In this paper we introduce a 3-phase steady state circuit formulation that is based on solving for complex AC currents and voltages in Cartesian form as the state variables. This is an extension of the single-phase equivalent circuit based approach proposed in [1]. The equivalent circuit of the 3-phase power grid is formed and split into real and imaginary sub-circuits to facilitate the use of the Newton-Raphson method for solving the resulting nonlinear circuit equations.

This physical model based approach to analyzing the grid enables decades of circuit simulation research to be applied to solve power flow problems. Most components in a distribution system, including induction motors, can be modeled as linear circuit elements, thereby resulting in fast and robust convergence. Moreover, the proposed simulation environment is capable of incorporating complicated nonlinear models for accurate power flow analysis of any imaginable smart-grid components. It, in turn, offers the opportunity of using the same unified models for different power system simulations (e.g., AC power flow, transient simulation, etc.). We will further show that our proposed formulation readily models short and open circuits in the event of failures for contingency analyses.

II. Background

An equivalent split circuit formulation of the power flow problem with current and voltage state variables was recently introduced in [1]. It is shown that the buses, transmission lines and other power system devices could be replaced with circuit elements (voltage sources, impedances, etc.). A key insight is that the equivalent circuit could itself be split into two sub-circuits: one real, and one imaginary, coupled by controlled sources. By splitting the circuit, its equations are no longer complex and, hence, the Newton Raphson (NR) method can be used to solve the nonlinear equations with fast convergence.

A graph-theory-based method known as tree-link analysis (TLA) is used to formulate the circuit equations for voltages and currents [5]-[6]. While Nodal Analysis formulation is applicable, with tree-link circuit analysis the resulting

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equations are more robust, and it is trivial to model both short- and open-circuit elements. This can enable more efficient contingency analysis, where it may be necessary to simulate a short or open between any two nodes in the event of a failure. Replacing lines with shorts or opens does not require the entire problem to be reformulated; and only local changes to the tree are required.

This tree-link formulation more effectively accommodates inductances and mutual couplings, thereby making it a preferred method for modeling the 3-phase power flow problem for distribution systems that are proposed in this paper.

III. Split Circuit Model

In this section we describe the mathematical and circuit models for some of the most critical system components.

A. Slack Bus Model

In the distribution system analysis, the transmission grid is usually modeled as a generator connected to the substation that feeds the power into the distribution system. This generator or slack bus is the simplest bus type to model. Depending on the configuration to which it is connected, in the real circuit (real portion of the split circuit [1]) it appears as an independent voltage source of value $|V_A|\cos\theta_A$, and in the imaginary circuit (imaginary portion of the split circuit [1]) it appears as a voltage source of value $|V_A|\sin\theta_A$. It should be noted that if the slack bus is connected in a Wye configuration, its magnitude $|V_A|$ represents the line-to-neutral voltage, while if connected as a delta configuration will represent the line-to-line voltage. The complete split circuit model for a 3-phase slack bus connected as a grounded Wye configuration is shown in Figure 1.

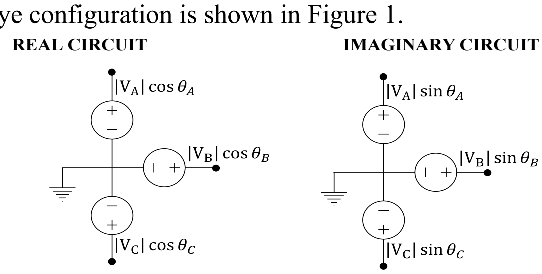

Figure 1: Complete split circuit of a slack bus generator.

B. Transmission Line Model

There are two possible ways of modeling the transmission line. The first approach is based on the Kron reduction [4], which eliminates the neutral line from the model. The other approach is by considering all four lines without any reduction.

After performing the Kron reduction, the transmission line branch currents are governed by Ohm's Law, where $\tilde{V}_{Aa}$, $\tilde{V}_{Bb}$ and $\tilde{V}_{Cc}$ are the voltage drops across the lines:

$$\begin{bmatrix}\tilde{I}_A\\ \tilde{I}_B\\ \tilde{I}_C\end{bmatrix}=\begin{bmatrix}\tilde{Y}_{aa} & \tilde{Y}_{ab} & \tilde{Y}_{ac}\\ \tilde{Y}_{ba} & \tilde{Y}_{bb} & \tilde{Y}_{bc}\\ \tilde{Y}_{ca} & \tilde{Y}_{cb} & \tilde{Y}_{cc}\end{bmatrix}\begin{bmatrix}\tilde{V}_{Aa}\\ \tilde{V}_{Bb}\\ \tilde{V}_{Cc}\end{bmatrix} \tag{1}$$

Since the admittances of the branches have both real and imaginary components ($Y_{ij}=\frac{1}{R_{ij}+jX_{ij}}=\frac{R_{ij}}{{R_{ij}}^2+{X_{ij}}^2}-j\frac{X_{ij}}{{R_{ij}}^2+{X_{ij}}^2}=G_{ij}^s+jB_{ij}^s$), the system from (1) can be split as:

$$\begin{bmatrix}I_A^R\\ I_A^I\\ I_B^R\\ I_B^I\\ I_C^R\\ I_C^I\end{bmatrix}=\begin{bmatrix}G_{aa}^s & -B_{aa}^s & G_{ab}^s & -B_{ab}^s & G_{ac}^s & -B_{ac}^s\\ B_{aa}^s & G_{aa}^s & B_{ab}^s & G_{ab}^s & B_{ac}^s & G_{ac}^s\\ G_{ba}^s & -B_{ba}^s & G_{bb}^s & -B_{bb}^s & G_{bc}^s & -B_{bc}^s\\ B_{ba}^s & G_{ba}^s & B_{bb}^s & G_{bb}^s & B_{bc}^s & G_{bc}^s\\ G_{ca}^s & -B_{ca}^s & G_{cb}^s & -B_{cb}^s & G_{cc}^s & -B_{cc}^s\\ B_{ca}^s & G_{ca}^s & B_{cb}^s & G_{cb}^s & B_{cc}^s & G_{cc}^s\end{bmatrix}\begin{bmatrix}V_{Aa}^R\\ V_{Aa}^I\\ V_{Bb}^R\\ V_{Bb}^I\\ V_{Cc}^R\\ V_{Cc}^I\end{bmatrix} \tag{2}$$

where the "R" and "I" superscripts denote the real and imaginary parts respectively.

Using the same approach, the transmission line shunt current can be written in the same way, where $\tilde{V}_A$, $\tilde{V}_B$ and $\tilde{V}_C$ are the line-to-neutral nodal voltages. Since the admittance of the shunt elements in the pi-model is purely imaginary ($\tilde{Y}_{ij}^{sh}=j\frac{B_{ij}}{2}=jB_{ij}^{sh}$), we derive the following formulation from Ohm's law:

$$\begin{bmatrix}I_{Ash}^R\\ I_{Ash}^I\\ I_{Bsh}^R\\ I_{Bsh}^I\\ I_{Csh}^R\\ I_{Csh}^I\end{bmatrix}=\begin{bmatrix}0 & -B_{aa}^{sh} & 0 & -B_{ab}^{sh} & 0 & -B_{ac}^{sh}\\ B_{aa}^{sh} & 0 & B_{ab}^{sh} & 0 & B_{ac}^{sh} & 0\\ 0 & -B_{ba}^{sh} & 0 & -B_{bb}^{sh} & 0 & -B_{bc}^{sh}\\ B_{ba}^{sh} & 0 & B_{bb}^{sh} & 0 & B_{bc}^{sh} & 0\\ 0 & -B_{ca}^{sh} & 0 & -B_{cb}^{sh} & 0 & -B_{cc}^{sh}\\ B_{ca}^{sh} & 0 & B_{cb}^{sh} & 0 & B_{cc}^{sh} & 0\end{bmatrix}\begin{bmatrix}V_A^R\\ V_A^I\\ V_B^R\\ V_B^I\\ V_C^R\\ V_C^I\end{bmatrix} \tag{3}$$

Equations (2) and (3) model the transmission line by using linear resistors and voltage-controlled current sources. Figure 2 further shows the proposed real part of the split circuit model for one of the three phases of a transmission line. The imaginary part of the split circuit as well as the split circuits for two other phases can be obtained in the same way.

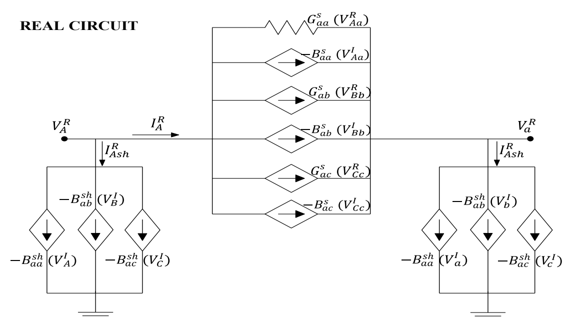

Figure 2: Real part of the split circuit of a transmission line (Phase A).

An alternative approach for transmission line modeling is to consider all four lines without Kron reduction. A similar split circuit model can be derived following the aforementioned steps where the neutral line is modeled as:

$$\tilde{V}_N=\tilde{Z}_{NA}\tilde{I}_A+\tilde{Z}_{NB}\tilde{I}_B+\tilde{Z}_{NC}\tilde{I}_C+\tilde{Z}_N\tilde{I}_N \tag{4}$$

Each impedance term has both real and imaginary parts, i.e. ($Z_{ij}=R_{ij}+jX_{ij}$). Substituting Z_{ij} into (4) yields:

$$V_N^R=R_N I_N^R-X_N I_N^I+R_{NA}I_A^R-X_{NA}I_A^I+R_{NB}I_B^R-X_{NB}I_B^I+R_{NC}I_C^R-X_{NC}I_C^I \tag{5}$$

$$V_N^I = R_N I_N^I + X_N I_N^R + X_{NA} I_A^R + R_{NA} I_A^I + X_{NB} I_B^R + R_{NB} I_B^I + X_{NC} I_C^R + R_{NC} I_C^I \quad (6)$$

The complete split circuit model of a neutral line is shown in Figure 3.

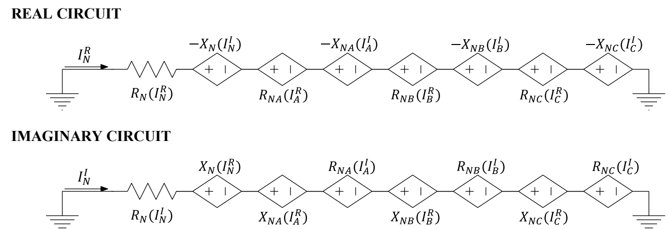

Figure 3: Complete split circuit model of a neutral line.

C. Induction Motor Model

An induction motor (IM) that operates under unbalanced conditions is traditionally modeled by an iterative symmetrical component model. At each iteration, the phase motor voltage is converted into sequence quantities, from which the positive and negative sequence currents are calculated and converted back into phase quantities [9]. In [4], an equivalent three-phase asymmetric impedance matrix is used to model the line-to-line voltages and the line currents for a specific slip:

$$\begin{bmatrix} \tilde{V}_{AB} \\ \tilde{V}_{BC} \\ \tilde{V}_{CA} \end{bmatrix} = \begin{bmatrix} \tilde{Z}_{AA} & \tilde{Z}_{AB} & \tilde{Z}_{AC} \\ \tilde{Z}_{BA} & \tilde{Z}_{BB} & \tilde{Z}_{BC} \\ \tilde{Z}_{CA} & \tilde{Z}_{CB} & \tilde{Z}_{CC} \end{bmatrix} \begin{bmatrix} \tilde{I}_A \\ \tilde{I}_B \\ \tilde{I}_C \end{bmatrix} \quad (7)$$

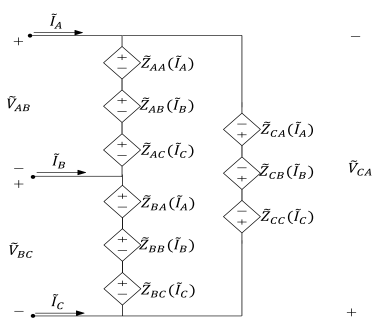

Figure 4: Unnatural circuit model of an induction motor.

This mathematically derived model, however, is known to produce an ill-conditioned impedance matrix that can result in numerical problems for power flow analysis. The genesis of this problem can be recognized from the physical representation of the corresponding circuit. Most notably, the model in (7) corresponds to an equivalent circuit that contains a loop of controlled voltage sources, as shown in Figure 4.

Any loop of ideal voltage sources is problematic for an equivalent circuit model since the current flowing through that loop is unbounded. To address this problem, we derive a new model that follows Kirchhoff's voltage law (KVL):

$$\tilde{V}_{AB} + \tilde{V}_{BC} + \tilde{V}_{CA} = 0 \quad (8)$$

Using Gaussian elimination, the linear system in (7) can be reduced to:

$$\begin{bmatrix} \tilde{V}_{AB} \\ \tilde{V}_{BC} \end{bmatrix} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} \begin{bmatrix} \tilde{I}_A \\ \tilde{I}_B \end{bmatrix} \quad (9)$$

It should be noted that this corresponds to the removal of the three controlled voltage sources in Figure 4 that are redundant.

We derive our proposed split circuit model for an induction motor by splitting (9) into real and imaginary parts:

$$\begin{bmatrix} V_{AB}^R \\ V_{AB}^I \\ V_{BC}^R \\ V_{BC}^I \end{bmatrix} = \begin{bmatrix} R_{11} & -X_{11} & R_{12} & -X_{12} \\ X_{11} & R_{11} & X_{12} & R_{12} \\ R_{21} & -X_{21} & R_{22} & -X_{22} \\ X_{21} & R_{21} & X_{22} & R_{22} \end{bmatrix} \begin{bmatrix} I_A^R \\ I_A^I \\ I_B^R \\ I_B^I \end{bmatrix} \quad (10)$$

Equation (10) models the induction motor by using linear resistors and current-controlled voltage sources. Figure 5 further shows the circuit schematic of our proposed linear model for a 3-phase induction motor.

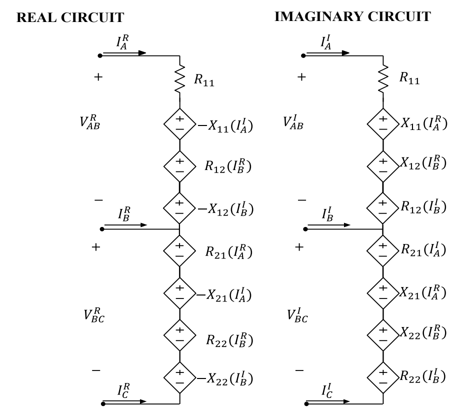

Figure 5: Equivalent split circuit model of an induction motor.

D. Constant PQ Load Model

A nonlinear PQ load model was derived in [1], where the real and imaginary load currents are represented as nonlinear functions of the real and imaginary bus voltages:

$$I_{RL} = \frac{P_L V_L^R + Q_L V_L^I}{(V_L^R)^2 + (V_L^I)^2} \quad (11)$$

$$I_{IL} = \frac{P_L V_L^I - Q_L V_L^R}{(V_L^R)^2 + (V_L^I)^2} \quad (12)$$

The load model in (11)-(12) can be directly applied to each phase of a 3-phase distribution system.

E. Constant Impedance Load Model

For the constant impedance load model, the real and reactive powers (P_0 and Q_0) are specified for the nominal voltage V_{L0}. The mathematical expression for modeling the bus voltage and load current is given in (13) and (14):

$$P_0 + jQ_0 = \tilde{V}_{L0} \left(\frac{\tilde{V}_{L0}}{\tilde{Z}_0} \right)^* \quad (13)$$

$$\tilde{I}_L = \frac{\tilde{V}_{L0}}{\tilde{Z}_0} \quad (14)$$

Solving (13) for $\tilde{Z}_0$, substituting it into (14), and splitting the real and imaginary parts yields:

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$$I_L^R = \frac{P_0}{|V_{L0}|^2} V_L^R + \frac{Q_0}{|V_{L0}|^2} V_L^I \quad (15)$$

$$I_L^I = \frac{-Q_0}{|V_{L0}|^2} V_L^R + \frac{P_0}{|V_{L0}|^2} V_L^I \quad (16)$$

Equations (15) and (16) model the constant impedance load by using an equivalent circuit with linear resistors and voltage -controlled current sources. The equivalent split circuit of an open Wye connected constant impedance load is shown in Figure 6.

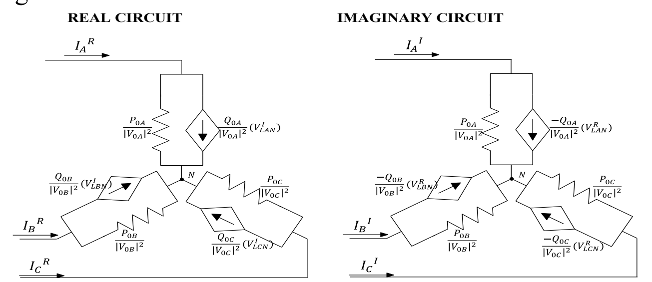

Figure 6: Equivalent split circuit model of an Open Wye connected constant impedance load.

F. Ideal Center-Tapped Transformer Model

The model for a standard transformer model was derived in [1]. However, a center-tapped transformer, as a special type of transformer, can be found as a branch element connecting buses in nearly every distribution network. We derive the split circuit model of a center-tapped transformer by relating the primary voltage with the secondary voltage that is tapped ($\tilde{V}_A$ and $\tilde{V}_a = \tilde{V}_{an} + \tilde{V}_{nb}$) through the turn ratios t_{an} and t_{bn} and the phase angle θ (which is only non-zero for phase shifters) [10]:

$$\frac{\tilde{V}_{an}}{\tilde{V}_A} = \frac{1}{t_{an}} e^{-j\theta} \quad (17)$$

$$\frac{\tilde{V}_{nb}}{\tilde{V}_A} = \frac{1}{t_{bn}} e^{-j\theta} \quad (18)$$

After we solve the real and imaginary parts for both $\tilde{V}_{an}$ and $\tilde{V}_{nb}$, we obtain the following secondary voltage expressions:

$$\begin{bmatrix} V_{an}^R \\ V_{an}^I \\ V_{nb}^R \\ V_{nb}^I \end{bmatrix} = \begin{bmatrix} \frac{\cos\theta}{t_{an}} & -\frac{\sin\theta}{t_{an}} & \frac{\cos\theta}{t_{bn}} & -\frac{\sin\theta}{t_{bn}} \\ \frac{\sin\theta}{t_{an}} & \frac{\cos\theta}{t_{an}} & \frac{\sin\theta}{t_{bn}} & \frac{\cos\theta}{t_{bn}} \end{bmatrix}^T \begin{bmatrix} V_A^R \\ V_A^I \end{bmatrix} \quad (19)$$

As can be seen from (19), each of the secondary voltages can be modeled with two voltage-controlled voltage sources that are controlled by the primary voltages in the real and imaginary circuits respectively. We can express the primary and secondary currents in terms of the turn ratios t_{an} and t_{bn}:

$$\tilde{I}_A = -\tilde{I}_{an} \frac{1}{t_{an}} e^{j\theta} - \tilde{I}_{nb} \frac{1}{t_{bn}} e^{j\theta} \quad (20)$$

Splitting (20) into real and imaginary parts, we obtain the following expressions for real and imaginary primary currents:

$$\begin{bmatrix} I_A^R \\ I_A^I \end{bmatrix} = \begin{bmatrix} \frac{-\cos\theta}{t_{an}} & \frac{\sin\theta}{t_{an}} & \frac{-\cos\theta}{t_{bn}} & \frac{\sin\theta}{t_{bn}} \\ \frac{-\sin\theta}{t_{an}} & \frac{-\cos\theta}{t_{an}} & \frac{-\sin\theta}{t_{bn}} & \frac{-\cos\theta}{t_{bn}} \end{bmatrix} \begin{bmatrix} I_{an}^R \\ I_{an}^I \\ I_{nb}^R \\ I_{nb}^I \end{bmatrix} \quad (21)$$

The primary currents can be modeled with four current-controlled current sources that are controlled by the currents I_{an}^R, I_{an}^I, I_{nb}^R and I_{nb}^I from the secondary side. The complete split circuit model is shown in Figure 7.

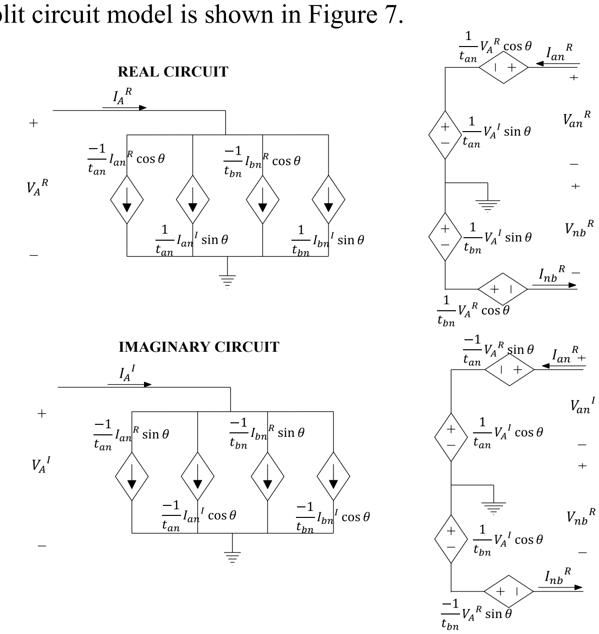

Figure 7: Equivalent split circuit of ideal center-tapped transformer.

IV. Results and Discussion

The equivalent circuit models derived in the previous section were applied to the IEEE 4-bus Wye–Delta center-tapped transformer case, as well as the regular IEEE 4-bus test cases. The 4-bus Wye-Delta center-tapped case is considered as an extremely challenging case in the literature [7] because the transformer connection, also known as the "4-wire delta" bank, is nontrivial to handle. Grounding the center tap shifts the secondary voltage reference to an unusual location for three-phase circuit analysis. It also results in unbalanced voltages and currents that can affect the three-phase motors and overload the transformer [9].

The schematic diagram of the Wye-Delta center tapped transformer 4-bus system with labeled elements is shown in Figure 8. Its real part of the equivalent split circuit model is shown in Figure 9. The imaginary part is symmetric and can be obtained on the same way as the real part. Using the description provided in [7], two configurations of the aforementioned system were considered. Since the proposed TLA formulation is capable of handling shorts and opens easily, we have implemented ideal switches in our test case (Figure 9). As such, by closing switch 1 and leaving switch 2 open, we obtain the first of two configurations. This configuration represents the normally operating 4-bus system with unbalanced loading and the ungrounded wye-delta transformer bank with a center-tapped transformer in one leg of the delta secondary. The second configuration can be obtained by reversing switches 1 and 2, representing a system that operates in the event of a failure (primary phase C fault).

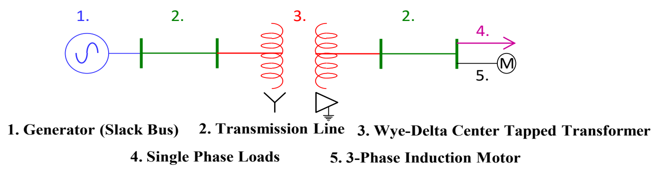

Figure 8: Schematic diagram of 4-bus test case.

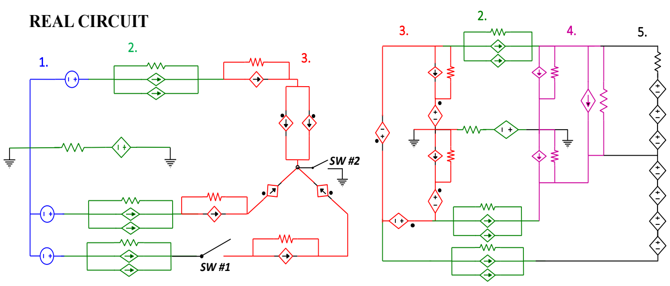

Figure 9: Real part of the split circuit model of the 4-bus test case.

Our prototype circuit solver was implemented in MATLAB. A graph and spanning tree were built and the TLA equations were formulated for the 4-bus test case. The TLA equations were solved using Newton-Raphson. The proposed implementation successfully simulated all aforementioned IEEE 4-bus test cases, as well as those reported in [7],[11]. The solutions for three-phase voltages at load bus are shown in Table 1 for various transformer and load configurations. Depending on load configuration at bus four, i.e. wye or delta, the three phase voltages are reported as phase or line voltages. Note that the solutions generated by the proposed method are in good agreement with the accepted solutions, also shown in Table 1.

Table 1: Results for load bus voltages (bus 4) for various configurations.

Configuration	Proposed Method [V∠°]	Results in [7],[11] [V∠°]
Balanced step-down Gr. Y-Gr.Y	V_A: 1918∠-9.1 V_B: 2062∠-128.3 V_C: 1981∠110.9	V_A: 1918∠-9.1 V_B: 2061∠-128.3 V_C: 1981∠110.9
Balanced step-up D-D	V_{AB}: 23659∠26.6 V_{BC}: 23690∠-93.5 V_{CA}: 23627∠146.5	V_{AB}: 23657∠26.6 V_{BC}: 23688∠-93.5 V_{CA}: 23625∠146.5
Balanced step-up D-Gr. Y	V_A: 13654∠26.6 V_B: 13679∠-93.5 V_C: 13645∠146.5	V_A: 13653∠26.6 V_B: 13678∠-93.5 V_C: 13644∠146.5
Unbalanced step-down Y-D (center-tapped), with IM	V_{AB}: 232∠-0.4 V_{BC}: 233∠-119.8 V_{CA}: 235∠119.5	V_{AB}: 232∠-0.5 V_{BC}: 233∠-119.8 V_{CA}: 235∠119.5
Unbalanced step-down open Y-D (center-tapped), with IM	V_{AB}: 231∠-0.3 V_{BC}: 233∠-120.7 V_{CA}: 231∠119.0	V_{AB}: 231∠-0.3 V_{BC}: 233∠-120.8 V_{CA}: 231∠118.9

Since most circuit elements in our proposed split circuit model are linear, the Newton-Raphson method converges quickly (after the second iteration). It is important to note that the induction motor model does not have to be solved iteratively like the traditional sequence model, and is simply modeled as a combination of linear circuit elements here.

V. Conclusion and Future Work

We have extended the recently introduced equivalent circuit formulation in [1] to the 3-phase steady state analysis of distribution power grids. Our preliminary results demonstrate that the proposed approach provides fast and robust convergence, and is not limited to balanced loads, particular network configurations, type of simulation etc. The proposed equivalent circuit and TLA approach have the ability to incorporate any electrical load (e.g. converters, solar cells, high voltage DC components, etc.) effortlessly into its formulation. Furthermore, the proposed approach in this paper allows for use of unified modeling methodology to perform various power system simulations such as steady-state power flow, transient, and contingency analysis on a given network. Toward future work, we intend to extend this simulation approach to perform steady-state and transient simulations on a given network without altering the equivalent circuit models between the two. This will be a significant improvement over the present commercially available methods, which require significant changes to the models if the transient analysis is performed on the network initially modeled for steady-state analysis.

Acknowledgment

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