Qualitative Design of Nonlinear Stochastic Filters

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Abstract

The paper discusses the qualitative behavior of nonlinear filters on the basis of the first order probability density function (pdf) of the error process, $p(\varepsilon)$. As an alternative to the usual error-norm-based criteria, such as L_1 or L_2 (mean square error), a new class of design criteria is proposed, which penalizes the discrepancies between the error density and a predefined function. We show that special forms of the criteria lead to commonly used performance measures. Finally, we introduce an approximation to the minimum mean square error for a given nonlinear problem.

Design Criteria

The criteria herein introduced are based on the fact that for a given nonlinear filtering problem there is a bound on the error entropy, H_ε. Let H_x be the entropy of the process to be estimated, $\mathcal{Z}$ represent the observation path, and $I(.,.)$ be the mutual information, then it is easy to show that, for any filter,

$$H_\varepsilon \geq H_x - I(x, \mathcal{Z}). \tag{1}$$

In addition, we can show the following converse to Shannon's maximum entropy result [4]. Let, H, be a fixed value for the entropy, then there exists a probability density function $p^*(\varepsilon)$, that minimizes the cost function,

$$D = \int_{-\infty}^{+\infty} d(\varepsilon) p(\varepsilon) d\varepsilon, \tag{2}$$

over all choices of pdf's $p(\varepsilon)$, where $d(\varepsilon)$ is a nonnegative distortion function. The minimizing $p^*(\varepsilon)$ is

$$p^*(\varepsilon) = \frac{\exp(\frac{d(\varepsilon)}{\gamma})}{\int_{-\infty}^{+\infty} \exp(\frac{d(z)}{\gamma}) dz} \tag{3}$$

where γ is a constant. In general, the existence of a particular filter that achieves the entropy bound and produces an error with density $p^*(\varepsilon)$, cannot be proved. It follows, as a natural requirement, that a filter should be designed having in mind simultaneously the error entropy and the error pdf.

To achieve this goal we introduce, [1], the functional

$$C_p(q) = H(q) + \mathcal{D}_K(q, p), \tag{4}$$

where $H(q)$ is the entropy associated with the pdf q, and $\mathcal{D}_K(q,p)$ is the Kullback directed divergence [2] between the functions q and p. The optimal filter is defined as the one that minimizes $C_p(q)$ for a given function p. It must be noticed that $\mathcal{D}_K(q,p) \geq 0$ with equality iff $q = p$. For each function p, the filter minimizes the sum of its error entropy and the pseudo-distance, $\mathcal{D}_K$, between its error pdf and p. The cost function $C_p(q)$ can be seen as a qualitative measure of performance in that it weights the shape of the error pdf.

For particular choices of the function p, the criterion $C_p(q)$ is shown to be equivalent to commonly used performance measures. For example, the minimum mean square error (MMSE) is obtained if p is a Gaussian function, leading to the interpretation of the MMSE optimal filter as the one that "tries" to produce a Gaussian error, while reducing the error entropy. In fact, minimizing D in (2) for a fixed entropy H and for $d(\varepsilon) = \varepsilon^2$ results in a Gaussian error density function with variance parameter

$$\widehat{\varepsilon^2} = (1/2\pi)\exp(2H - 1). \tag{5}$$

In [3] and [1], experiments for the absolute phase demodulation problem, show that a Gaussian type error density is obtained, if the MMSE strategy is used to compute the estimate. Specializing for linear problems where the MMSE optimal filter's error is Gaussian, the new criteria says that minimizing the mean square error is equivalent to minimizing the error entropy. A similar result can be stated for the L_1 norm where the function p, in $C_p(q)$, is taken as the Laplacian density.

MMSE Approximation

We observed that for an important class of nonlinear problems, alternative filters exhibited similar error entropies. We use this and the above ideas to derive an approximation to the MMSE associated with a given nonlinear problem. If H^+ is the error entropy associated with a sub-optimal filter, and using (5), we write the following expression for the minimum mean square error

$$\widehat{\varepsilon^2_{\min}} \simeq (1/2\pi)\exp(2H^+ - 1). \tag{6}$$

The expression provides a computationally inexpensive method to approximate the mean square error of the optimal filter. Results are shown in [1].

References

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