Sample Path Description of Gauss Markov Random Fields [1]

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Abstract — **We provide a characterization of Gauss Markov random fields in terms of partial differential equations with random forcing term. Our method consists of obtaining a concrete representation of an abstract stochastic partial differential equation using some results from the theory of vector measures.**

I. Preliminary Background

To fix notations, let $Y_u, u \in K$ (K compact subset of R^n) be a random field and let D be an open subset of K. Let Γ be the boundary of D. Then it is well known that the field Y_u is Markov with respect to D if

$$\mathrm{E}[Y_u Y_v \mid \sigma(\Gamma)] = \mathrm{E}[Y_u \mid \sigma(\Gamma)]\mathrm{E}[Y_v \mid \sigma(\Gamma)] \tag{1}$$

where $u \in D$ and $v \in D^c$ and $\sigma(\Gamma)$ is the usual germ-field given by

$$\sigma(\Gamma) = \cap\{\sigma(O) : O \text{ open and } O \supset \Gamma\} \tag{2}$$

Thus D and D^c are conditionally independent given knowledge of the boundary.

For Gaussian fields , conditioning on $\sigma(A)$ ($A \subset K$) is projection onto the closed subspace generated by $Y_u, u \in A$, instead of the larger subspace of all L^2 functions measurable with respect to $\sigma(A)$ (see [2]). Hence, the Markov property can be formulated in terms of projection on these smaller Hilbert spaces. We introduce the spaces

$$H(K) = \text{closed subspace generated by } Y_u, u \in K \tag{3}$$

and the corresponding reproducing kernel Hilbert space $\mathcal{H}(K)$. It is well known that $H(K)$ and $\mathcal{H}(K)$ are isometrically isomorphic through the mapping, $J\colon H(K) \mapsto \mathcal{H}(K)$

$$JY(t) = \mathrm{E}YY_t \; t \in K. \tag{4}$$

II. Sample Path Characterization

We assume that $C_0^\infty(K)$ is dense in $\mathcal{H}(K)$. For $u, v \in C_0^\infty(K)$, we can write the inner product of $\mathcal{H}(K)$ in the form

$$< u, v >_{\mathcal{H}(K)} = (Pu, v)_{L^2} \tag{5}$$

where P is a differential operator written in the divergence form (see [3]).

In order to derive a sample path characterization we use the well known technique for associating a generalized random field ξ to our ordinary random field Y through the following formula

$$\xi(\phi) = \int_K Y(u,\omega)\phi(u)\, du. \tag{6}$$

A generalized random field can be regarded as a linear operator from C_0^∞ to a space of L^2 random variables. With every generalized field, there is an associated dual field. The dual field ξ^* is also a linear operator from C_0^∞ to L^2 random variables such that

$$\mathrm{E}[\xi(u)\xi^*(v)] = \int_K u(t)v(t)\, dt \tag{7}$$

Kallianpur and Mandrekar [5] has shown that an ordinary random field is Markov if and only if the associated generalized random field is Markov. This enables us to study the generalized random field and then transfer back its properties to the associated ordinary random field.

Now, the generalized field is Markov if the dual field ξ^* is local, (see [1]) in the sense that if $\mathrm{supp}u \cap \mathrm{supp}v = \phi$, then $\mathrm{E}(\xi^*(u)\xi^*(v)) = 0$. Locality of the dual field implies that $\mathrm{E}[\xi^*(u)\xi^*(v)] = (Pu, v)_{L^2}$ where P is the same differential operator associated with inner product of the RKHS of the ordinary random field (see equation (5)).

We know [1] that ξ satisfies the following abstract equation

$$\xi(Pu) = \xi^*(u). \tag{8}$$

We show that the dual of the generalized field is intimately related to J^{-1} (J is defined in equation (4)). Under some integrability condition we further show that the mapping J^{-1} is weakly compact. A weakly compact mapping from $L^1(\mu)$ to a Hilbert space is Riesz representable (see [4]). Therefore, we can write equation (8) in the following weak form

$$\int Y(t,\omega)Pu(t)\, dt = \int \epsilon(t,\omega)u(t)\, dt \tag{9}$$

where $u(t) \in C_0^\infty(K)$.

When the support of u is in D, a subset of K, we relate $\xi^*(u)$ to minimum mean square error. In particular, we show that $\xi^*(u)$ lies in the closed subspace generated by $(Y_u - \mathrm{E}[Y_u \mid \sigma(\Gamma)]), u \in D$. This provides a canonical description which is analogous to the one provided by Woods [6] in the context of Gauss Markov random fields on lattices.

References

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