GAP DETECTOR FOR MULTIPATH

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ABSTRACT

In a multipath communication channel, the optimal receiver is matched to the maximum likelihood (ML) estimate of the multipath signal. In general, this leads to a computationally intensive multi-dimensional nonlinear optimization problem. In this paper, we develop a detection algorithm that avoids the ML estimation while still achieving good performance. Our approach is based on a geometric interpretation of the problem. The ML estimate of the multipath signal is the orthogonal projection of the received signal on a suitable signal subspace $\mathcal{S}$. We design a second subspace $\mathcal{G}$, the representation subspace, that is close to $\mathcal{S}$, but whose orthogonal projection is easily computed. The "closeness" is measured by the gap metric. The subspace $\mathcal{G}$ is designed by using wavelet multiresolution analysis tools coupled with a reshaping algorithm in the Zak transform domain. We show an example where our approach significantly outperforms the correlator receiver and an alternative suboptimal approach.

1. INTRODUCTION

In a multipath communication channel, the detection problem can be cast as follows

$$H_1 : r(t) = s_m(t) + n(t) \quad (1)$$
$$H_0 : r(t) = n(t) \quad (2)$$

where the multipath signal $s_m(t)$ is

$$s_m(t) = \sum_{k=1}^{K} \alpha_k s(t - \tau_k) \quad (3)$$

The transmitted signal $s(t)$ is assumed to be known. The number of paths K, the attenuation factors $\{\alpha_k\}$, and the delays $\{\tau_k\}$ are all unknown parameters. For simplicity, we assume that the additive noise $n(t)$ is white and Gaussian.

For a known signal $s(t)$, a simple detector is the correlator receiver which correlates the received signal $r(t)$ with the transmitted signal $s(t)$, and uses the peaks in the correlator output to estimate and detect the multipath signal. This method is simple and easy to implement. If different returns of the transmitted signal are separated in time by more than the duration of the signal autocorrelation function, the correlator receiver is equivalent to the optimal generalized likelihood ratio test (GLRT) receiver. Unfortunately, this condition is not satisfied in many practical situations. When the condition is not satisfied, the correlation method is not optimal.

On the other hand, the optimal GLRT receiver requires the ML estimates of the unknown channel parameters. This involves a multi-dimensional nonlinear optimization over these parameters. If K is large and delayed replicas of the transmitted signal overlap, as in the shallow water channel, the GLRT receiver is out of reach.

In this paper, we explore an alternative approach which preserves the simplicity of the correlator receiver while exhibiting the performance of the optimal GLRT receiver. The approach is based on a geometric interpretation of the multipath detection problem. We observe that the multipath noise free signal $s_m(t)$ forms a multipath signal subspace $\mathcal{S}$. The GLRT statistic is only a function of the orthogonal projection of $r(t)$ on $\mathcal{S}$. However, finding this orthogonal projection directly is difficult because, as mentioned before, it requires a multi-dimensional nonlinear optimization over all the $\{\tau_k\}$. Instead, we approximate the optimal orthogonal projection by projecting $r(t)$ onto another subspace $\mathcal{G}$, a representation subspace [1], [2], whose orthogonal projection is easily computed and which remains close to the multipath signal subspace $\mathcal{S}$. The "closeness" between subspaces is measured by the gap metric. The subspace $\mathcal{G}$ is designed in two steps: the first step designs $\mathcal{G}$ as the subspace close, in the gap sense, to the integer shift signal subspace $\mathcal{S}_{\text{int}}$; and the second step reshapes the generating function $g(t)$ of $\mathcal{G}$ to be as shiftable as possible [3]. Simulation results have shown that, in the presence of severe multipath, the minimum gap receiver provides about 3.4dB gain over the correlator receiver and 4dB gain over a multiple replica integer shift receiver.

2. GEOMETRIC INTERPRETATION AND GAP METRIC

We start by reviewing the geometric interpretation of the multipath detection problem proposed by [1]. We introduce the multipath signal subspace

$$\mathcal{S} = \left\{ s_m(t : \gamma, K) = \sum_{k=1}^{K} \alpha_k s(t - \tau_k), K \in \mathbb{Z}^+, \alpha_k, \tau_k \in \mathbb{R} \right\} \quad (4)$$

which is the collection of all possible multipath signals. The parameter γ is the set of channel parameters $\{(\alpha_k, \tau_k)\}$. The GLRT statistic for the detection problem given in (1)

This work was partially supported by DARPA through ONR grant N00014-91-J183

and (2) is

$$L = \| P_S(r(t)) \|_2^2. \quad (5)$$

where $\| \cdot \|_2$ is the L_2 norm and $P_S(r(t))$ is the orthogonal projection of $r(t)$ on $\mathcal{S}$

$$P_S(r(t)) = s_m(t : \gamma^*, K^*) \quad (6)$$

where γ^*, K^* are the ML estimates of the unknown channel parameters

$$(\gamma^*, K^*) = \arg \min_{(\gamma,K)} \| r(t) - s_m(t : \gamma, K) \|_2^2 \quad (7)$$

Minimizing (7) is a difficult multi-dimensional nonlinear optimization problem because $s_m(t)$ is a nonlinear function of the delays $\{\tau_k\}$. It is desirable to find a computationally fast algorithm that avoids the explicit computation of the unknown parameters. This is the goal of this paper. In our approach, we design an alternative subspace, the representation subspace $\mathcal{G}$, whose orthogonal projection is easily computed and which is close to the multipath signal subspace $\mathcal{S}$. The "closeness" between subspaces is measured by the gap metric.

The gap metric [4] is a distance measure between two closed subspaces. Given two closed subspaces $\mathcal{S}$ and $\mathcal{G}$ in a Hilbert space H, we denote by $\mathcal{S}_S$ the unit sphere of $\mathcal{S}$ (the set of all $u \in \mathcal{S}$ with $\| u \|_2 = 1$) and let

$$\hat{\delta}(\mathcal{S}, \mathcal{G}) = \sup_{u \in \mathcal{S}_S} \text{dist}(u, \mathcal{G}) \quad (8)$$

where $\text{dist}(u, \mathcal{G}) = \inf_{v \in \mathcal{G}} \| u - v \|_2$. Likewise, we define $\hat{\delta}(\mathcal{G}, \mathcal{S})$. The quantity

$$\delta(\mathcal{S}, \mathcal{G}) = \max(\hat{\delta}(\mathcal{S}, \mathcal{G}), \hat{\delta}(\mathcal{G}, \mathcal{S})) \quad (9)$$

is called the gap between $\mathcal{S}$ and $\mathcal{G}$.

An equivalent perhaps more intuitive definition of the gap metric is also given in [4]. It is defined in terms of the orthogonal projection operators of $\mathcal{S}$ and $\mathcal{G}$. Denote by P_S and $P_{\mathcal{G}}$ the orthogonal projection operators of $\mathcal{S}$ and $\mathcal{G}$ respectively, then the gap metric is

$$\delta(\mathcal{S}, \mathcal{G}) = \| P_S - P_{\mathcal{G}} \| \quad (10)$$

here $\| \cdot \|$ is the L_2 induced operator norm.

As we mentioned early, our goal is to find a representation subspace $\mathcal{G}$ such that the orthogonal projection on $\mathcal{G}$ is close to the orthogonal projection on $\mathcal{S}$. From equation (10), we can achieve this goal by minimizing the gap between $\mathcal{S}$ and $\mathcal{G}$ because if the gap is small, $P_{\mathcal{G}}$ is close to P_S in the norm sense, then we can approximate $P_S(r(t))$ by $P_{\mathcal{G}}(r(t))$.

Once the representation subspace $\mathcal{G}$ has been designed, we approximate the GLRT statistic by using the orthogonal projection of the received signal $r(t)$ on $\mathcal{G}$, i.e., by

$$\tilde{L} = \| P_{\mathcal{G}}(r(t)) \|_2^2 \quad (11)$$

Since each element in $\mathcal{S}$ is a linear combination of different delays of the transmitted signal $s(t)$, it is intuitive to design the representation subspace in a similar way. We choose as the representation subspace

$$\mathcal{G} = \left\{ \sum_{n=-\infty}^{+\infty} \beta_n g(t-n), \beta_n \in \mathbb{R} \right\} \quad (12)$$

i.e., $\mathcal{G}$ is spanned by the integer delayed replicas of a single function $g(t)$. The design of the subspace $\mathcal{G}$ is now reduced to the design of its generator, the function $g(t)$. We assume that $\{g(t-n), n \in \mathbb{Z}\}$ is a Riesz basis of $\mathcal{G}$, [5], i.e., $\exists\ A, B$ such that $0 < A \leq B < \infty$ and

$$A \leq \sum_n |\mathcal{F}_g(f+n)|^2 \leq B \quad \text{a.e.} \quad (13)$$

where $\mathcal{F}_g(f)$ is the Fourier transform of $g(t)$. With this assumption, $P_{\mathcal{G}}(r(t))$ is *easily* obtained by calculating the inner product of $r(t)$ with integer shifts of the biorthogonal function $\tilde{g}(t)$ of $g(t)$, thus, avoiding the multi-dimensional nonlinear optimization.

3. SUBSPACE DESIGN

In this section, we discuss the computation and optimization of the gap metric. Our goal is to find a representation subspace $\mathcal{G}$ in the form given by (12) that minimizes the gap $\delta(\mathcal{S}, \mathcal{G})$ between $\mathcal{S}$ and $\mathcal{G}$. From the definition of the gap

$$\hat{\delta}(\mathcal{S}, \mathcal{G}) = \sup_{\| s_m \|_2 = 1} \ \inf_{g_m(t) \in \mathcal{G}} \| s_m(t) - g_m(t) \|_2 \quad (14)$$

where $s_m(t)$ is the multipath signal in (3) and $g_m(t) = \sum_n \beta_n g(t-n)$. Calculating $\hat{\delta}(\mathcal{S}, \mathcal{G})$ directly is not an easy task. Since $s_m(t)$ is a nonlinear function of $\{\tau_k\}$, taking the supremum over $\| s_m \|_2 = 1$ requires a multi-dimensional nonlinear optimization which is what precisely we are trying to avoid. The major reason for this difficulty is with the optimization over arbitrary real valued $\{\tau_k\}$ in equation (14). To circumvent the problem, we design $\mathcal{G}$ in two steps. We first introduce an integer shift signal subspace $\mathcal{S}_{\text{int}}$

$$\mathcal{S}_{\text{int}} = \left\{ \sum_{n=-\infty}^{+\infty} \alpha_n s(t-n), \alpha_n \in \mathbb{R} \right\} \quad (15)$$

We then design $\mathcal{G}$ to be close to $\mathcal{S}_{\text{int}}$ in the sense of the gap metric. Secondly, we use the reshaping algorithm in Benno and Moura [3] to reshape the generating function $g(t)$ of $\mathcal{G}$ so that $g(t)$ is as translation invariant, or shiftable, as possible, see [3] for details. A function $g(t)$ is shiftable if

$$g(t-\tau) = \sum_n \beta_n g(t-n) \quad \forall \tau \in [0,1) \quad (16)$$

What equation (16) says is that if a function is shiftable, then any arbitrary delay of the function is well represented by a linear combination of the integer shifts of the same function. The goal is for the new nearly shiftable function and its integer shifts to represent well not only a linear combination of the integer shifts of $s(t)$, but also its arbitrary real valued shifts.

The first step has been reduced to finding a linear subspace $\mathcal{G}^*$ such that the gap $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$ between $\mathcal{S}_{\text{int}}$ and $\mathcal{G}$ is minimized. In the following, we also assume that $\{s(t-n), n \in \mathbb{Z}\}$ is a Riesz basis for $\mathcal{S}_{\text{int}}$.

We need an explicit formula for the gap $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$ between $\mathcal{S}_{\text{int}}$ and $\mathcal{G}$. Theorem 1 provides this.

Theorem 1 *Let $\mathcal{S}_{\text{int}} = \{\sum \alpha_n s(t-n)\}$, $\mathcal{G} = \{\sum \beta_n g(t-n)\}$ be two closed linear subspaces, where $\{s(t-n)\}$ and $\{g(t-n)\}$ satisfy the Riesz basis condition in (13). Then, the square of the gap between $\mathcal{S}_{\text{int}}$ and $\mathcal{G}$ is given by*

$$\delta^2(\mathcal{S}_{\text{int}}, \mathcal{G}) = 1 - \inf_{f\in[0,1)} \frac{|\sum_n \mathcal{F}_s(f+n)\overline{\mathcal{F}_g(f+n)}|^2}{\sum_n |\mathcal{F}_s(f+n)|^2 \sum_n |\mathcal{F}_g(f+n)|^2} \tag{17}$$

where $\mathcal{F}_s(f)$ and $\mathcal{F}_g(f)$ are the Fourier transforms of $s(t)$ and $g(t)$. $\overline{\mathcal{F}_g(f)}$ is the complex conjugate of $\mathcal{F}_g(f)$. The infimum is taken over the regions where the function is continuous.

Theorem 1 is used to find the optimal subspace $\mathcal{G}^*$ that minimizes $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$, i.e.,

$$\mathcal{G}^* = \arg\min_{\mathcal{G}} \delta(\mathcal{S}_{\text{int}}, \mathcal{G}) \tag{18}$$

Equivalently,

$$g^*(t) = \arg\max_{g(t)} \inf_{f\in[0,1)} \frac{|\sum_n \mathcal{F}_s(f+n)\overline{\mathcal{F}_g(f+n)}|^2}{\sum_n |\mathcal{F}_s(f+n)|^2 \sum_n |\mathcal{F}_g(f+n)|^2} \tag{19}$$

To perform the optimization over $g(t)$ in (19), we restrict the search of the function $g(t)$ to the set of compactly supported orthonormal scaling functions of a multiresolution analysis. There are three major reasons to do this. First, it is clear that if $g(t) = s(t)$, then the gap in (17) is zero. However, $s(t)$ is in general not shiftable. After reshaping $s(t)$ using the algorithm in [3] to make it nearly shiftable, we have observed that the subspace spanned by the integer shifts of the new reshaped signal does not approximate the subspace $\mathcal{S}_{\text{int}}$ well. On the other hand, from our simulations, the reshaped orthonormal scaling functions do still approximate well $\mathcal{S}_{\text{int}}$. Secondly, if we look at $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$ carefully, we notice that, for a fixed value of $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$, there are many choices of $g(t)$ because $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$ is only related to the minimum of

$$\frac{|\sum_n \mathcal{F}_s(f+n)\overline{\mathcal{F}_g(f+n)}|^2}{\sum_n |\mathcal{F}_s(f+n)|^2 \sum_n |\mathcal{F}_g(f+n)|^2} \tag{20}$$

We use this additional freedom to require the minimizer $g(t)$ to be a compactly supported orthonormal scaling function. Finally, compactly supported orthonormal scaling functions are nicely parameterized, [5], [6]. Using these parameterizations of the function $g(t, \theta)$, the optimization of $\delta(\mathcal{S}_{\text{int}}, \mathcal{G})$ is done by a search over the parameter space of θ

$$\theta^* = \arg\max_{\theta} \inf_{f\in[0,1)} \frac{|\sum_n \mathcal{F}_s(f+n)\overline{\mathcal{F}_g(\theta, f+n)}|^2}{\sum_n |\mathcal{F}_s(f+n)|^2 \sum_n |\mathcal{F}_g(\theta, f+n)|^2} \tag{21}$$

where $\theta = [\theta_1, \cdots, \theta_N]^T$ is the vector of parameters defining the compactly supported orthonormal scaling functions and N is the number of unconstrained parameters [6]. The optimal scaling function is given by

$$g^*(t) = g(t, \theta^*) \tag{22}$$

The corresponding representation subspace $\mathcal{G}^*$ represents $\mathcal{S}_{\text{int}}$ well. The second step of our approach considers the arbitrary shifts of $s(t)$. We use the reshaping algorithm in [3] which essentially reshapes the Zak transform of the energy density function of $g^*(t)$ to make it nearly shiftable.

4. SIMULATION RESULTS

In this section, we illustrate the use of the detector described above with a numerical example.

We choose a modulated decaying exponential as the transmitted signal $s(t)$

$$s(t) = \exp(-t) \cdot \cos(t) \cdot u(t) \tag{23}$$

where $u(t)$ is the unit step function. The number of paths K is set to 15. We set all attenuation factors $\{\alpha_k\}$ to be equal to 1. The delays $\{\tau_k, k = 1, \cdots, 15\}$ are generated by a random number generator. The test statistic

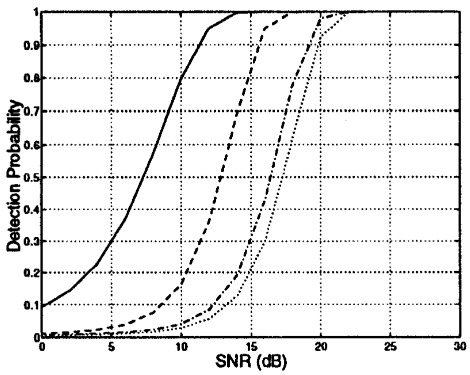
$$\tilde{L} = \| P_{\mathcal{G}^*}(r(t)) \|_2^2 \tag{24}$$

is chi-square distributed under H_0 and noncentral chi-square distributed under H_1 [7].

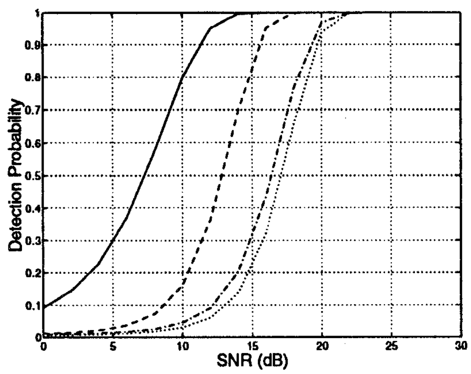
We compare the performance of our new receiver with two other receivers. The first receiver that we compare with is the correlator receiver. Secondly, we compare our detector with a "matched filter with integer shifts" (MFIS) detector. The MFIS matches the received signal with integer shifts of $s(t)$. It provides an approximation of the GLRT detector without the large penalty in computational effort when the number of paths K is large. We did not compute the performance of the optimal GLRT receiver because of its high computational cost of optimization over a 15-dimensional parameter space.

Fig.1(a)-(d) show the detection probability P_D as a function of the SNR for different multipath delay patterns. The false alarm probability P_F is fixed at 0.01. There are 4 sets of curves in the figures. The solid lines represent the ideal matched filter which are obtained by assuming the multipath signal $s_m(t)$ is fully known, i.e., the parameters K, $\{\alpha_k\}$ and $\{\tau_k\}$ are all known. Since in practice, the channel parameters are not known, the solid lines provide an over optimistic bound for the performance. The performance of the optimal GLRT receiver (which is not computed) will degrade the performance of the ideal matched filter. The dashed lines are the minimum gap receiver we have designed. The dashdotted and the dotted lines represent the correlator receiver and the MFIS receiver respectively.

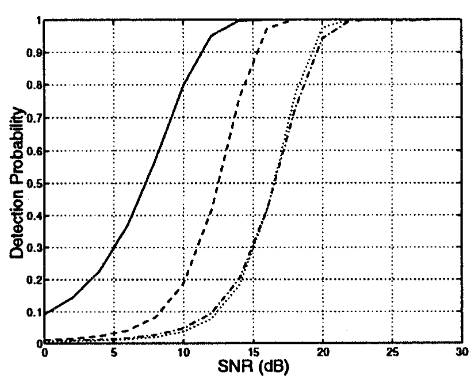
Analysis of Fig.1(a)-(d) shows that the minimum gap receiver provides an average gain of about 3.4dB over the correlator receiver and a gain of about 4dB over the MFIS. The reason for the gain over the correlator receiver is that, in our example, different delayed replicas of the transmitted signal overlap, so the correlator receiver is not optimal. The reason for the gain over MFIS is that $\sum_n \beta_n g^*(t-n)$ not only matches well with linear combinations of integer shifts of $s(t)$, but also with linear combinations of arbitrary shifts of $s(t)$ while the MFIS only matches with linear combinations of the integer shifts. The figures also show that the performance of our approach is almost the same for different delay patterns. The dashed lines in the figures that represent the minimum gap detector performance are practically coincident. In other words, the gap detector is robust to the multipath distortion. On the contrary, the performance of MFIS varies with the delay patterns. There is about a 1.5dB difference between Fig.1(a) and Fig.1(d).



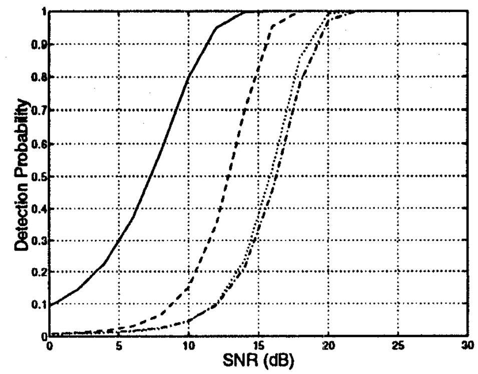
(a)



(b)



(c)



(d)

Figure.1 ROC curves, the solid line is the ideal matched filter, "– –" is the minimum gap receiver, "–·" is the correlator receiver and "· · ·" is the MFIS receiver ($K = 15$, $P_F = 0.01$). (a) Delay pattern 1. (b) Delay pattern 2. (c) Delay pattern 3. (d) Delay pattern 4.

5. SUMMARY

This paper develops a shiftable minimum gap detector that is fine tuned to multipath detection. We design a representation subspace $\mathcal{G}$ that is matched to the multipath signal subspace $\mathcal{S}$ in the gap sense. The minimum gap receiver is simple to implement. It avoids the multiple-dimensional optimization required by the optimal receiver while providing about 3.4dB gain over the simple correlator receiver and 4dB gain over the multiple replica integer shift receiver.

6. REFERENCES

[1] S. A. Benno and J. M. F. Moura, "Scaling functions optimally robust to translations," *to appear in Multidimensional Systems & Signal Processing*, 1996.

[2] S. A. Benno and J. M. F. Moura, "On translation invariant subspaces and critially sampled wavelet transforms," *submitted*, May 1995.

[3] S. A. Benno and J. M. F. Moura, "Nearly shiftable scaling functions," in *ICASSP*, pp. II–1097–1100, May 1995.

[4] T. Kato, *Perturbation theory for linear operators.* Springer Verlag, 2nd ed., 1976.

[5] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Commun. Pure and Appl. Math.*, vol. 41, pp. 909–996, Nov. 1988.

[6] H. Zou and A. H. Tewfik, "Parameterization of compactly supported orthonormal wavelets," *IEEE Trans. Signal Processing*, vol. 41, pp. 1428–1431, March 1993.

[7] B. Friedlander and B. Porat, "Detection of transient signals by the Gabor representation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 169–179, Feb. 1989.