TIME DELAY DETERMINATION: MAXIMUM LIKELIHOOD AND KALMAN-BUCY TYPE STRUCTURES

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ABSTRACT

Time delay determination is an important problem in numerous applications. The approach taken here models the signals via linear differential equations driven by white noise. The time delays are unknown parameters modulating the received signals. The maximum likelihood estimation of the delays requires the filtering in the minimum mean square error (MMSE) sense of the signals. The problem becomes that of the joint estimation of the signals with the identification of the delays. Due to the structure of the signal model, the signal MMSE estimate is obtained via a recursive structure of the Kalman-Bucy type.

The class of signals considered includes the stationary signals, to which the cross-correlation receivers are restricted. In fact, it can be shown that the receiver studied in this paper is a generalization of the cross-correlation receiver. The paper presents the general receiver structure, discussing it in the context of a specific example. The Cramer-Rao bound associated with the delay estimation is also discussed.

INTRODUCTION

The signal received by the array of sensors is

$$z(t,s) = y(t-D_a(s)) + v(t,s), \quad t\in T, \; s\in S, \tag{1}$$

where t is the time parameter, s is the space variable that takes values on a set of coordinates S ($S=\{1,2,\dots,S\}$), and $D_a(s)$ represents the actual delay. The observation noise $v(t,s)$ is modeled as a Gaussian white noise vector process with covariance matrix $R(t_1,s_1,s_2)\delta(t_1-t_2)$. The signal $y(t)$ is

$$\frac{dx(t)}{dt}=A(t)x(t)+B(t)u(t), \tag{2.a}$$

$$y(t)=C(t)x(t), \qquad t\geq t_0; \tag{2.b}$$

the state $x(t)$ is a real vector with initial condition $x(t_0)$ which is a Gaussian random vector with mean $\bar{x}(t_0)$ and covariance matrix $\Sigma(t_0)$; the dynamics disturbance $u(t)$ is modeled as a Gaussian white noise vector process, independent of the process $v(t,s)$ and of the random vector $x(t_0)$, with covariance matrix $Q(t_1)\delta(t_1-t_2)$.

Depending on the system matrices $A(t)$, $B(t)$, $C(t)$, equations (2) model nonstationary narrow or broadband processes. The above model includes as a special case the stationary signals.

TIME DELAY ESTIMATOR

The time delay estimator is developed via maximum likelihood techniques. From [1], the delay vector estimate is given by

$$\hat{D}^S(t)= \min_{D^S} J(t,t_i;D^S) \tag{3.a}$$

with the log-likelihood function

$$J(t,t_i;D^S)=\int_{t_i}^{t}\{[Z(r)-\hat{Y}(r,D^S)]^T R^{-1}(r)[Z(r)-\hat{Y}(r,D^S)] + \text{tr}[R^{-1}(r)P_Y(r,D^S)]\}dr \tag{3.b}$$

where

$$Z(t)=[z^T(t,1)|z^T(t,2)|\dots|z^T(t,S)]^T$$

$$D^S=[D(1)|D(2)|\dots|D(S)]^T$$

$$R(t)= \begin{bmatrix} R(t,1,1) & R(t,1,2) & \dots & R(t,1,S) \\ R(t,2,1) & R(t,2,2) & \dots & R(t,2,S) \\ \vdots & \vdots & \ddots & \vdots \\ R(t,S,1) & R(t,S,2) & \dots & R(t,S,S) \end{bmatrix}.$$

The conditional mean signal estimate $\hat{Y}(t,D^S)$ and the error covariance matrix $P_Y(t,D^S)$ appearing in (3.b) are given by

$$\hat{Y}(t,D^S)= \sum_{s=1}^{S} C_s(t-D(s))\hat{x}(t,D(s)|D^S) \tag{4}$$

$$P_Y(t,D^S)= \sum_{s=1}^{S}\sum_{m=1}^{S} C_s(t-D(s)).P(t,D(s),D(m)|D^S).C_m^T(t-D(m)) \tag{5}$$

where

$$C_s(t)=[0|\dots|0|C^T(t)|0|\dots|0]^T$$

$$\hat{x}(t,a|D^S)=E[x(t-a)|Z^t,D^S]$$

$$P(t,a_1,a_2|D^S)=E\{[x(t-a_1)-\hat{x}(t,a_1|D^S)].[x(t-a_2)-\hat{x}(t,a_2|D^S)]^T|D^S\}$$

$$a\geq D_{min};\; D_{min}=\min\{D(s),\;\forall s\in S\}$$

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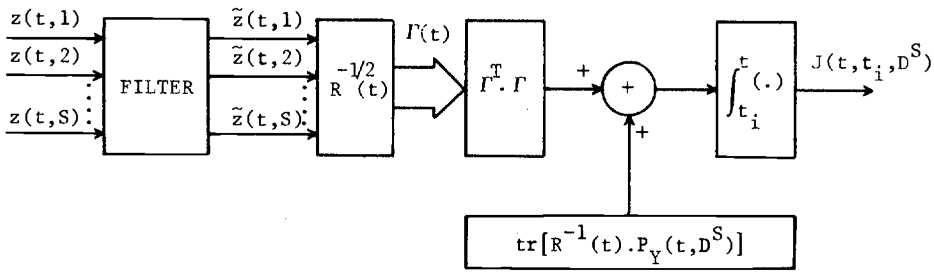

Figure 1: Delay Estimator Structure

Figure 1 illustrates the delay estimator structure. The filter block accomplishes the minimum mean square error (MMSE) signal estimate conditioned on the delay vector. It represents a Kalman-Bucy type filter developed for systems with delays in the observations [3]. The state vector estimate $\hat{x}(t,a|D^S)$ is given by the partial differential equation [1]

$$\frac{\partial\hat{x}(t,a|D^S)}{\partial t}+\frac{\partial\hat{x}(t,a|D^S)}{\partial a}=K(t,a|D^S).[Z(t)-\hat{Y}(t,D^S)] \tag{6.a}$$

with the boundary condition

$$\frac{d\hat{x}(t,D_{min}|D^S)}{dt}=A(t-D_{min}).\hat{x}(t,D_{min}|D^S)+K(t,D_{min}|D^S).[Z(t)-\hat{Y}(t,D^S)] \tag{6.b}$$

where

$$K(t,a|D^S)=[\sum_{s=1}^{S}P(t,a,D(s)|D^S)C_s^T(t-D(s))].R^{-1}(t)$$

represents the filter gain matrix. The state estimate error covariance matrix $P(t,a_1,a_2|D^S)$ is given by coupled partial differential equations, see [1] for details. These partial differential equations represent the extension to filtering with delays in the observations of the Riccati equation associated with the Kalman-Bucy filters.

CRAMER-RAO BOUND

For any unbiased estimator, the Cramer-Rao bound is given by the inverse of the Fisher information matrix $I(Z^t,D_a^S)$ [4]-[6], i.e.,

$$E\{[\hat{D}^S(t)-D_a^S].[\hat{D}^S(t)-D_a^S]^T\}\geq I^{-1}(Z^t,D_a^S). \tag{7}$$

The elements $I_{ij}(Z^t,D_a^S)$ are given by

$$I_{ij}(Z^t,D_a^S)=-E\{\frac{\partial^2\ln[p(Z^t|D^S)]}{\partial D(i)\partial D(j)}\Big|_{D^S=D_a^S}\} \tag{8}$$

where $p(Z^t|D^S)$ is the probability density function of the observations given the delays.

It can be shown [2] that,

$$I_{ij}(Z^t,D_a^S)=\int_{t_i}^{t}tr\{R^{-1}(r)[P_{Y_D}(r,D_a^S)_{ij}+\overline{\hat{Y}_D(r,D_a^S)_i}.\overline{\hat{Y}_D(r,D_a^S)_j}^T+\frac{1}{2}.\frac{\partial^2P_Y(t,D^S)}{\partial D(i)\partial D(j)}\Big|_{D^S=D_a^S}]\}dr \tag{9}$$

where

$$\hat{Y}_D(t,D^S)_i=\partial\hat{Y}(t,D^S)/\partial D(i)$$

$$\overline{\hat{Y}_D(t,D^S)_i}=E[\hat{Y}_D(t,D^S)_i]$$

$$P_{Y_D}(t,D^S)_{ij}=E\{[\hat{Y}_D(t,D^S)_i-\overline{\hat{Y}_D(t,D^S)_i}].[\hat{Y}_D(t,D^S)_j-\overline{\hat{Y}_D(t,D^S)_j}]^T\}.$$

From (4)

$$\hat{Y}_D(t,D^S)_i=\sum_{s=1}^{S}C_s(t-D(s)).\hat{\eta}_i(t,D(s)|D^S)+C_i(t-D(i)).\hat{\gamma}(t,D(i)|D^S)+[\partial C_i(t-D(i))/\partial D(i)].\hat{x}(t,D(i)|D^S) \tag{10}$$

where

$$\hat{\eta}_i(t,a|D^S)=\partial\hat{x}(t,a|D^S)/\partial D(i)$$

$$\hat{\gamma}(t,a|D^S)=\partial\hat{x}(t,a|D^S)/\partial a.$$

From equation (10) it follows that $\hat{Y}_D(t,D^S)_i$ and $\hat{Y}_D(t,D^S)_j$ are the output of a dynamic system with state vector consisting of $x(t)$, $\hat{x}(t,a|D^S)$, $\hat{\gamma}(t,a|D^S)$, $\hat{\eta}_i(t,a|D^S)$, and $\hat{\eta}_j(t,a|D^S)$ with white noise inputs $u(t)$ and $v(t,s)$.

A CASE STUDY

Consider the problem of estimating the time delay between the signals observed at two spatially separated sensors ($S=\{1,2\}$). The observations are modeled by equation (1) where we assume $D_a(1)=0.$ and $D_a(2)=D_a=0.005$ seconds. The observation noise covariance matrix is $R(t_1,s_1,s_2).\delta(t_1-t_2)$ where

$$R(t_1, s_1, s_2) = \begin{cases} 0.1 & \text{if } s_1 = s_2 \\ 0. & \text{elsewhere} \end{cases}.$$

This covariance structure models spatially uncorrelated noise. The signal process y(t) is a nonstationary signal modeled as a first-order, linear dynamical system of the type of (2), where we assume C(t)=1. The state variable initial condition is zero mean with covariance $\Sigma(-0.014)=10.0$, and the dynamics disturbance covariance is Q=20.0. Functions A(t) and B(t) are given by

$$A(t)=-50.[\exp(-2t^2)+1]$$

$$B(t)=5.[1+0.1/\sqrt{0.01+t^2}]$$

Finally assume that one can restrain the delay domain to the time interval [-0.014, +0.014] seconds. Establishing a discretization step of 0.001 seconds, the time delay estimation processor is implemented through a bank of 29 Kalman-Bucy type filters working in parallel. Each filter is tuned to an allowable value of the delay (D=-0.014+0.001k; k=0,1,...,28). The log-likelihood function is built by means of these filters' output.

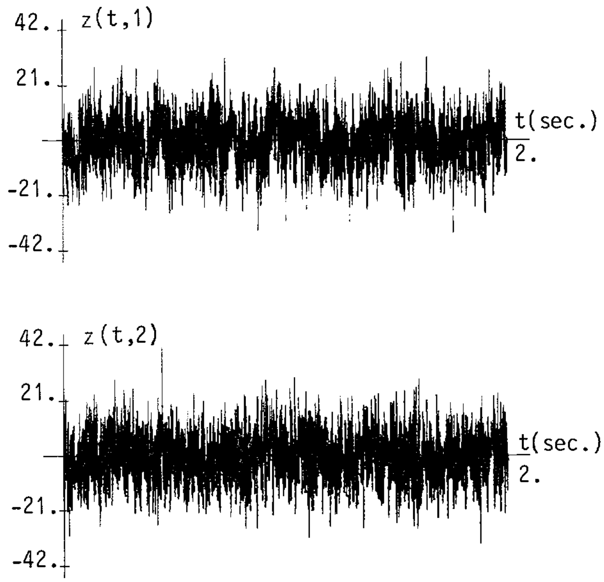

Figure 2: Observation Process

The simulation study was carried out using computer generated data. Figure 2 shows one sample of the two sensors observation process. Figure 3 illustrates the achieved delay estimate time evolution. One can see that, after a short transient, it settles down around the true delay.

In order to get the delay processor statistical behaviour, a 500 sample Monte Carlo experiment was run. The delay estimate ensemble average and the mean square error variance time evolutions are plotted in figures 4 and 5, respectively. One can see from the time evolution of the ensemble average that the delay estimate is asymptotically unbiased.

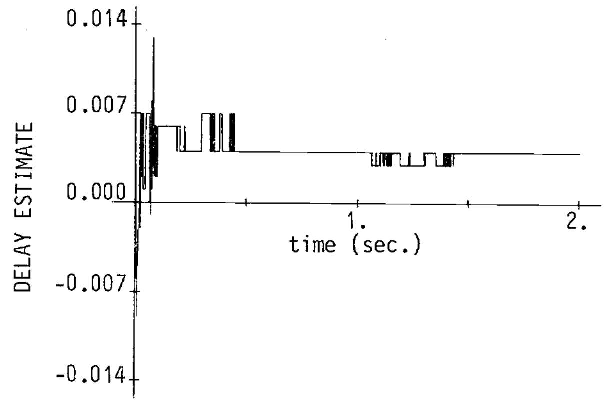

Figure 3: Delay Estimate

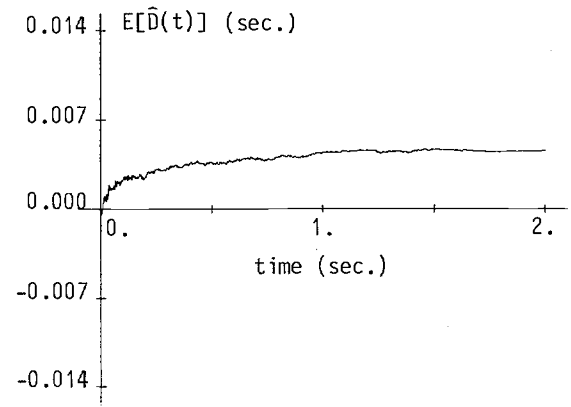

Figure 4: Delay Estimate Ensemble Average

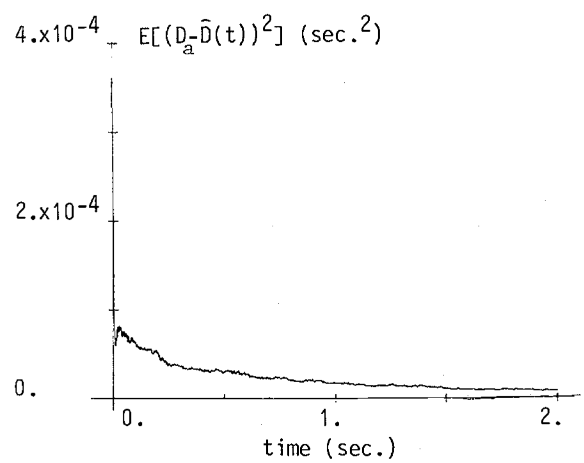

Figure 5: Mean Square Error Variance

Finally, figure 6 plots the Cramer-Rao lower bound computed from (7)-(10), see [2] for details. Figures 5 and 6 considered together clearly show that the time delay estimate mean square error variance achieves the Cramer-Rao lower bound.

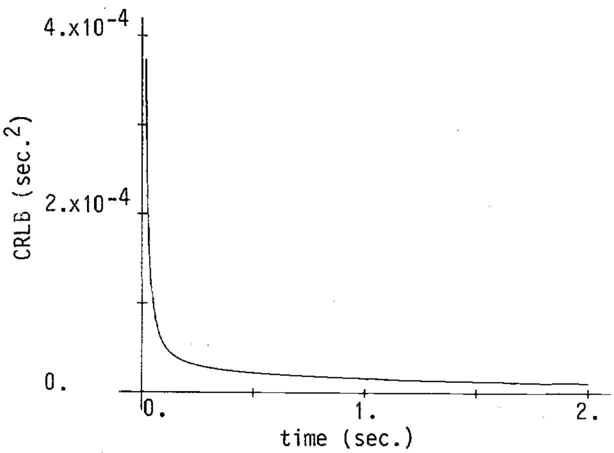

Figure 6: Cramer-Rao Bound

CONCLUSION

The work presented in this paper reports on maximum likelihood delay estimation with stochastic nonstationary signals. The delay processor presented performs the estimation of the signals via a recursive structure of the Kalman-Bucy type, along with the identification of the delays.

Maximum likelihood theory provides optimal estimates that are asymptotically unbiased (consistency) and that asymptotically achieve Cramer-Rao bound (efficiency). For the case study presented, the simulation results obtained are seen to follow closely these properties.

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