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# Ramanujan Topologies for Decision Making in Sensor Networks

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***Abstract*— We consider the design of the topology of the communication graph $G = (V, E)$ supporting distributed decision in sensor networks with $N = |V|$ sensors. The number $M$ of links connecting the sensors, i.e., the number of edges $|E| = M$ in the graph $G$, is fixed. We assume a simple binary decision test where the data may be spatially correlated. The global detector performs a threshold test on a weighted fusion of the local likelihood ratios, which can be computed in a distributed fashion using a consensus algorithm. The graph topology plays a central role in the convergence speed of the distributed detector. Exhaustive search over the class of possible communication networks is unrealistic. Our solution is constructive. We first reduce this topology design to a spectral graph optimization problem; specifically, to designing the topology that maximizes the ratio $\gamma$ of the algebraic connectivity to the largest eigenvalue of the graph Laplacian. Borrowing results from spectral graph theory, we show that for the class of non-bipartite Ramanujan graphs $\gamma \geq \gamma_{\text{min}}$. The importance of this inequality is that $\gamma_{\text{min}}$, asymptotically, is an upper bound on $\gamma$ for most classes of graphs. The paper discusses the commonly used explicit constructions of Ramanujan graphs and their impact on the convergence speed of distributed consensus. In particular, it shows that these graphs perform much better even for finite values of $N$ than highly structured networks, or small world type graphs, or Erdös-Renýi random networks.**

## I. Introduction

We consider the problem of distributed decision making in sensor networks with spatially correlated sensor measurements. Each sensor computes a local statistic based on the observed data. The final decision is computed by fusing all the local statistics. Distributed detection is a well-studied problem, and much work has been done on designing the optimal global fusion rule for different detection scenarios (see [1], [2], [3] for a detailed discussion on these.) Previous work on distributed detection assumes a parallel (centralized) or sequential decision fusion architecture, which is unrealistic in many applications. For example, a parallel architecture assumes the existence of a central hub to which all the local data are communicated directly and then a final decision is made. Such a fusion scheme is prone to failures of the fusion center or to communication bottlenecks that render it inoperative. Sequential architectures suffer from low convergence speed of the data fusion process, and is sensitive to failure of any intervening node. On the other hand, all-to-all connectivity, leading to a complete graph, may not be possible because of communication constraints. This discussion shows that the inter-sensor connectivity network is an important design issue. We study the problem of designing the optimal network topology (in the sense of the convergence speed of the decision fusion algorithm) when we constrain the number of inter-sensor communication links. In particular, we consider a simple binary hypothesis test with spatially correlated sensor measurements in a Gaussian environment. Under such circumstances, the optimal global detector performs a threshold test on a weighted average of the local (sensor) observations. Using a distributed consensus algorithm (see [4], [5]) for computing averages on a graph, we show that each local sensor iteratively achieves the performance of the optimal centralized detector. Thus, through local communication among the sensors (defined by the connectivity graph), we attain performance equivalent to the global detector using centralized fusion. The convergence speed of the detection algorithm is directly related to that of the consensus algorithm. We have studied previously the problem of designing the optimal network topology (under a given constraint on the number of links) leading to the fastest convergence rate of the consensus algorithm, see [6], [7], [8]. In this paper, we consider the case of correlated sensor measurements and show that the convergence properties of the local detector depends on the eigenratio $\gamma$ between the algebraic connectivity and the largest Laplacian eigenvalue of the connectivity network. Specifically, maximizing the convergence speed is equivalent to finding the network with largest $\gamma$, which meets the constraint on the number of links. Using results from spectral graph theory, we establish a significant lower bound on $\gamma$ for the class of non-bipartite Ramanujan graphs, and through various arguments and numerical studies show that these graphs are essentially optimal with respect to the convergence speed.

A brief outline of the rest of the paper follows. Section II summarizes elementary spectral graph theory concepts, needed for the development of the paper. Section III formulates the main distributed detection problem, while Sections IV and V relate the convergence speed to the eigenratio $\gamma$. Sections VI and VII discuss some results from spectral graph theory, which motivate the use of non-bipartite Ramanujan graphs.

Section VIII gives explicit constructions of Ramanujan graphs available in the literature, while Section IX presents numerical studies. Finally, Section X concludes the paper.

## II. Elementary Spectral Graph Theory

We define a graph $G = (V, E)$ as a 2-tuple, where $V$ denotes the set of $N$ vertices and $E$ the set of $M$ edges. There exists an edge between vertices $n$ and $l$, if they can communicate with each other directly, and denote it by the unordered pair $(n, l) \in E$. The connectivity pattern of the graph is given by an $N \times N$ symmetric matrix, called the adjacency matrix $A$, defined as

$$A_{n,l} = \begin{cases} 1 & \text{if } (n,l) \in E \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We define the neighborhood $\Omega_n$, of node $n$ as

$$\Omega_n = \{l \in V : \ (n,l) \in E\} \tag{2}$$

The degree of a node $n$ is the number of its neighbors and is given by

$$d_n = |\Omega_n| \tag{3}$$

We define the Laplacian $L$ of the graph as

$$L = D - A \tag{4}$$

where, $D = \text{diag}(d_1, ..., d_N)$. It can be shown that $L$ is a symmetric positive semidefinite matrix (see [9]), and hence we can arrange its non-negative eigenvalues as

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq ... \leq \lambda_N(L) \tag{5}$$

The multiplicity of the zero eigenvalue is equal to the number of connected components of $L$ and thus, for connected graphs, $\lambda_2(L) > 0$, see [9]. Unless otherwise stated, every graph in this paper is assumed to be connected.

## III. Problem Formulation

We consider a simple binary hypothesis testing problem with spatially correlated sensor measurements. Specifically, we assume a shift-in-mean Gaussian detection problem, with equal covariance matrices on both hypotheses. Thus, denoting by $\mathbf{y} \in \mathbb{R}^{N\times 1}$ the vector of sensor measurements, we have

$$\text{under } \mathcal{H}_p : \ \mathbf{y} = \mathbf{m}_p + \xi, \ p = 0, 1 \tag{6}$$

where we assume $\mathbf{m}_1 = -\mathbf{m}_0 = \mathbf{m}$ and

$$\xi \ \sim \ \mathcal{N}(\mathbf{0}, K) \tag{7}$$

where, $\mathbf{0} \in \mathbb{R}^{N\times 1}$ is the vector of all zeros and $K$ is a positive definite covariance matrix. For simplicity, we consider a minimum probability of error detection problem, with equal prior probabilities. It can be shown from standard detection theory (see [10]) that, in such a situation, the optimal global test is given by

$$l(\mathbf{y}) = \frac{2}{N}\mathbf{m}^T K^{-1}\mathbf{y} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} 0 \tag{8}$$

where, $l(\mathbf{y})$ is the sufficient global statistic. Define $\omega = [\omega_1 ... \omega_N]^T$ as

$$\omega^T = 2\mathbf{m}^T K^{-1} \tag{9}$$

Then, from eqn.(8), it follows that the global statistic can be written as an average of local statistics. In other words,

$$l(\mathbf{y}) = \frac{1}{N}\sum_{n=1}^{N} l_n(y_n) \tag{10}$$

where $l_n(y_n) = \omega_n y_n, \ i = 1, ..., N$, are local statistics.

## IV. Distributed Average Consensus Algorithm

Let $\mathbf{x}(0) \in \mathbb{R}^{N\times 1}$ be the vector of initial sensor measurements or states. We define the vector of averages as

$$\mathbf{x}_{\text{avg}} = \overline{r}\mathbf{1} \tag{11}$$

where $\mathbf{1} \in \mathbb{R}^{N\times 1}$ is the vector of ones and $\overline{r} = \frac{1}{N}\mathbf{1}^T\mathbf{x}(0)$. The distributed average consensus algorithm computes the average $\overline{r}$ at each sensor, starting from an initial state $\mathbf{x}(0) = [x_1(0)...x_N(0)]^T$, using linear distributed iterations of the form (see [4])

$$x_n(i+1) = W_{nn}x_n(i) + \sum_{l\in\Omega_n} W_{nl}x_l(i), \ n = \{1, ..., N\} \tag{12}$$

Collecting the $N$ equations in (12) in matrix-vector form

$$\mathbf{x}(i+1) = W\mathbf{x}(i) = W^{i+1}\mathbf{x}(0) \tag{13}$$

where the sparsity pattern of $W$ is determined by the underlying connectivity network. In particular, for $n \neq l$, $W_{nl} = 0$, if $(n, l) \notin E$.
The consensus algorithm converges if

$$\lim_{i\to\infty} \| \mathbf{x}(i) - \mathbf{x}_{\text{avg}} \|_2 = 0 \tag{14}$$

for any initial state vector $\mathbf{x}(0) \in \mathbb{R}^{N\times 1}$. In other words, convergence occurs if

$$\lim_{i\to\infty} W^i = \frac{1}{N}J \tag{15}$$

where $J = \mathbf{1}\mathbf{1}^T$. It turns out that, for a given network, the choice of the edge weights in eqn.(14) plays an important role in determining the convergence rate of the algorithm. We consider the case of optimum equal weights (see [4]) in this paper in which the weight matrix is

$$W = I - \alpha L \tag{16}$$

with

$$\alpha = \frac{2}{\lambda_2(L) + \lambda_N(L)} \tag{17}$$

where $L$ is the network Laplacian matrix (see [4] for other weight design techniques.)
It can be shown that (see [6]) with such a weight assignment we have

$$\| \mathbf{x}(i) - \mathbf{x}_{\text{avg}} \|_2 \ \leq \ \rho^i \| \mathbf{x}(0) - \mathbf{x}_{\text{avg}} \|_2 \tag{18}$$

where $\rho$ is given by

$$\rho = \frac{1 - \lambda_2(L)/\lambda_N(L)}{1 + \lambda_2(L)/\lambda_N(L)} = \frac{1-\gamma}{1+\gamma} \tag{19}$$

For connected graphs, $|\rho| < 1$ and we have convergence (see [6]). From eqns.(18 and 19) it follows that, for faster convergence, $\rho$ should be as small as possible, which in turn implies that $\gamma$ should be as large as possible. In other words,

$$\text{fast convergence } \Rightarrow \text{ small } \rho \Rightarrow \text{ large } \gamma \tag{20}$$

## V. Performance Bounds for Distributed Detection

From eqn.(10), we note that the global sufficient statistic is an average of local statistics. This means that the global statistic $l(\mathbf{y})$ can be computed using the distributed averaging algorithm. For this, we set the initial state vector to the local statistics, i.e., we start the consensus algorithm with

$$\mathbf{x}(0) = [l_1(y_1)...l_N(y_N)]^T \tag{21}$$

The state update is given by

$$\mathbf{x}(i+1) = W\mathbf{x}(i) \tag{22}$$

where $W$ is given by eqn.(16). At any time $i$, we consider the following test at sensor $n$,

$$x_n(i) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} 0 \tag{23}$$

where $x_n(i)$ is the $n$-th component of $\mathbf{x}(i)$. Let $P_e^n(i)$ denote the probability of error of the corresponding test. Also, let $P_e$ be the probability of error of the optimum global test (see eqn.(8).) Then, from Section IV, it follows

$$\lim_{i\to\infty} P_e^n(i) = P_e, \ n = 1, ..., N \tag{24}$$

The convergence speed of $P_e^n(i)$ is determined by the convergence speed of $x_n(i)$ to the global statistic $l(\mathbf{y})$, and thus by the factor $\gamma$.

For the simplified case of $\mathbf{m} = \mu\mathbf{1}$ and $K = \sigma^2 I$, we now establish a bound on $P_e^n(i)$ in terms of $\gamma$. It can be shown that (see [6])

$$P_e = \text{erfc}\left(\frac{\mu\sqrt{N}}{\sigma}\right) \tag{25}$$

where $\text{erfc}(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-z^2/2}dz$. Also, it can be shown that (see [6])

$$P_e \leq P_e^n(i) \leq \text{erfc}\left(\frac{\mu\sqrt{N}}{\sigma\sqrt{1+\rho^{2i}(N-1)}}\right) \tag{26}$$

where $\rho$ is given in eqn.(19). Eqn.(26) shows that the local detector achieves the performance of the optimal global detector if $\rho < 1$, and, further, that the smaller $\rho$ is, the faster the convergence. From eqn.(20), it follows that for fast convergence $\gamma = \lambda_2(L)/\lambda_N(L)$ should be as large as possible and hence the topology design problem may be stated as

$$\text{maximize } \gamma \tag{27}$$

where the maximization needs to be carried out over the class of networks meeting the given constraint on the number of links.

## VI. Spectrum of Regular graphs and Ramanujan graphs

In this section, we state a few results from spectral graph theory, which motivate the use of non-bipartite Ramanujan graphs as a candidate for optimal topology.

A regular graph $G$ with $N$ vertices and degree $k$ is a graph with number of vertices $N$, where all vertices have the same degree $k$. We now state some well-known facts about the spectrum of regular graphs. Let $G_{N,k}$ be a $k$-regular graph on $N$ vertices. Then, we can arrange the eigenvalues of its adjacency matrix $A$ as (see [11])

$$k = \lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_N(A) \geq -k \tag{28}$$

In particular, the multiplicity of the eigenvalue $k$ is equal to the number of connected components of the graph, and $\lambda_N(A) = -k$ *iff* the graph is bipartite (see [11].) From eqn.(28) it follows that

$$|\lambda_j(A)| \leq k, \ \forall j \in \{1, ..., N\} \tag{29}$$

It can be shown that, for $k$-regular graphs, the Laplacian eigenvalues (see eqn.(5)) are related to those of the adjacency matrix by

$$\lambda_j(L) = k - \lambda_i(A) \tag{30}$$

We call an eigenvalue of magnitude $k$ a trivial eigenvalue. We now state a well-known theorem by Alon and Boppana (see [12]).

*Theorem 1* Let $\{G_{N(m),k}\}_{m\geq 1}$ be a family of $k$-regular graphs, where the number of vertices $N(m) \to \infty$ as $m \to \infty$. Also, let $\lambda(A_m)$ denote the magnitude of the largest non-trivial eigenvalue of the adjacency matrix $A_m$ in absolute value. Then,

$$\liminf_{m\to\infty} \lambda(A_m) \geq 2\sqrt{k-1} \tag{31}$$

Thus, assuming that the limit exists, we have, for any family of $k$-regular graphs, where the number of vertices $N \to \infty$,

$$\lim_{N\to\infty} \lambda(A) \geq 2\sqrt{k-1} \tag{32}$$

A $k$-regular graph is called Ramanujan if (see [13])

$$\lambda(A) \leq 2\sqrt{k-1} \tag{33}$$

It follows from eqns.(28 and 30) that for a (connected) non-bipartite Ramanujan graph of degree $k$ (see [6])

$$\lambda_2(L) \geq k - 2\sqrt{k-1} \tag{34}$$

$$\lambda_N(L) \leq k + 2\sqrt{k-1} \tag{35}$$

Hence, for (connected) non-bipartite Ramanujan graphs,

$$\gamma = \frac{\lambda_2(L)}{\lambda_N(L)} \geq \frac{k - 2\sqrt{k-1}}{k + 2\sqrt{k-1}} \tag{36}$$

In the sequel, whenever we mention Ramanujan graphs, we actually refer to connected non-bipartite Ramanujan graphs. Also, as mentioned earlier, all the graphs considered in this paper are connected.

There exist explicit constructions of infinite families of non-bipartite Ramanujan graphs (see [13]), and each graph of the family satisfies the lower bound on $\gamma$ given in eqn.(36). It follows from eqn.(31) that this lower bound is in fact an asymptotic (in the number of nodes $N$) upper bound on $\gamma$ for families of $k$-regular graphs (see [6]), and hence the class of non-bipartite Ramanujan graphs are optimal asymptotically among the class of $k$-regular graphs. This suggests that the non-bipartite Ramanujan graphs are suitable candidates for topology design in large sensor networks.

## VII. Regular vs Non-Regular Graphs

In this section, we present an inequality relating $\gamma$ to the degree distribution of graphs, which shows that heterogeneity in the degree distribution does not favor large values of $\gamma$. We recall two results from spectral graph theory. For any graph $G$ with $N$ vertices, let us define

$$d_{\min} = \min(d_j,\ j \in \{1, ..., N\}) \tag{37}$$

where, $d_j$ is the degree of the $j$-th node (see eqn.(3).) Also,

$$d_{\max} = \max(d_j,\ j \in \{1, ..., N\}) \tag{38}$$

In other words, $d_{\min}$ and $d_{\max}$ are the minimum and maximum degrees of $G$ respectively. Then, we have (see [14])

$$\lambda_2(L) \leq \frac{N}{N-1} d_{\min} \tag{39}$$

and

$$\lambda_N(L) \geq \frac{N}{N-1} d_{\max} \tag{40}$$

From eqns.(39) and 40) it follows that, for any graph $G$,

$$\gamma = \frac{\lambda_2(L)}{\lambda_N(L)} \leq \frac{d_{\min}}{d_{\max}} \tag{41}$$

Eqn.(41) shows that, for graphs with large heterogeneity (large spread) in degree distribution, the value of $\gamma$ is small and hence such networks are not good from the point of view of consensus algorithms.

## VIII. Explicit Constructions of Ramanujan Graphs

In this section, we provide constructions of Ramanujan graphs available in the literature. Explicit constructions of infinite families of Ramanujan graphs exist for the case, where $k-1$ is a prime (see [13], [15]) or a prime power (see [16]). In this paper, we consider two constructions of non-bipartite Ramanujan graphs based on Lubotzky-Phillips-Sarnak (LPS) (see [13]), and call them LPS-I and LPS-II. First we summarize some concepts from algebra, needed for the development of the rest of the paper. We also briefly outline the procedure of Cayley graph construction, which provides a way of constructing regular graphs using group theory.

### A. Algebraic Concepts

We start with the definition of a group (see [17].)

*Definition 2 (Group)* : We define a group $X$ to be a non-empty set of elements, equipped with a binary operation ".", satisfying the following properties:

1) $a.b \in X, \quad \forall a, b \in X$ (closure property)
2) $a.(b.c) = (a.b).c, \quad \forall a, b, c \in X$ (associative property)
3) There exists an element $e \in X$, called the identity element, such that $a.e = e.a = a, \quad \forall a \in X$
4) For each $a \in X$, there exists an element $a^{-1} \in X$, called the inverse of $a$, such that $a.a^{-1} = a^{-1}.a = e$

We are now in a position to describe Cayley graphs as follows.

*Cayley Graphs*: We start with a group $X$, consisting of $N$ elements and a $k$-element symmetric subset, $S \subset X$ (by a symmetric subset we mean, $s \in S \ \Rightarrow \ s^{-1} \in S$.) The set $S$ is often called the set of generators in the literature. We now form a graph $G = (V, E)$ from $X$, by choosing the vertex set $V = X$, and $(u, v) \in E$ *iff* $vu^{-1} \in S$. It follows that the graph $G$, formed in this way, is $k$-regular (see [18]).

We now summarize some number theory concepts, required in the sequel.

*Definition 3 (Congruence)* : The statement $a \equiv b \bmod (n)$ implies that $a - b$ is divisible by $n$.

*Definition 4 (quadratic Residue)* : We call $a$ is a quadratic residue modulo $b$, if there exists an integer $c$, such that $c^2 \equiv a \bmod (b)$.

*Definition 5 (Legendre Symbol)* : We define the Legendre symbol $\left(\frac{a}{p}\right)$ for an integer $a$ and a prime $p$ as

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \text{ divides } a \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \end{cases} \tag{42}$$

For the LPS-I construction, we also need to describe the Projective Special Linear group PSL(2,$Z/qZ$).

*PSL(2,$Z/qZ$)*: The set $Z/qZ = \{0, 1, .., q-1\}$ is the field of integers modulo $q$, where $q$ is a prime. We start by considering the set of $2 \times 2$ matrices with entries from the field $Z/qZ$, such that determinants are non-zero quadratic residues modulo $q$. We then can define an equivalence relation on this set, where two matrices belong to the same equivalence class, if one is a non-zero scalar multiple of the other (here

scalar refers to an element of the field $Z/qZ$.) The set of all these equivalence classes is the group PSL$(2, Z/qZ)$, see [19].

We are now in a position to give the non-bipartite Ramanujan graph constructions as follows.

### B. LPS-I Graphs

Let us take two unequal primes $p$ and $q$, congruent to 1 modulo 4, such that the Legendre symbol $\left(\frac{p}{q}\right) = 1$. Let $X$ be the Projective Special Linear group, PSL(2,$Z/qZ$). It can be shown that $|X| = \frac{q(q^2-1)}{2}$ (see [13].) The LPS-I graph is a Cayley graph over the group $X$ of order $N = |X|$ and degree $k = p + 1$. The set of $p + 1$ generators for this construction is given as follows. We choose an integer $i$, such that $i^2 \equiv -1 \bmod (q)$ (the fact that $q$ is a prime congruent to 1 modulo 4 guarantees the existence of such an $i$.) Also, let $\beta = (a_0, a_1, a_2, a_3)$ be a solution of the equation of the diophantine equation

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p \tag{43}$$

It can be shown from a theorem by Jacobi (see [13]) that there are $p+1$ solutions of this equation, with $a_0 > 0$ and odd, and $a_j$ even for $j = 1, 2, 3$. To each such solution $\beta$ we assign a $2 \times 2$ matrix, $\widetilde{\beta}$, in PSL(2,$Z/qZ$) as

$$\widetilde{\beta} = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix} \tag{44}$$

These $p+1$ matrices form the generator set $S$, and the LPS-I graph is produced by the action of $S$ on PSL(2,$Z/qZ$). The graph $G$ produced in this way is a connected non-bipartite Ramanujan graph, with number of vertices $N = \frac{q(q^2-1)}{2}$ and degree $k = p + 1$. As an example of an LPS-I construction, we may choose $p = 13$ and $q = 17$. It follows that $p$ and $q$ are congruent to 1 modulo 4 and $\left(\frac{p}{q}\right) = 1$. In this case, we get a connected non-bipartite Ramanujan graph of degree $k = 14$ and order $N = 2448$.

The only drawback with the LPS-I graphs is that the number of vertices grows as $O(q^3)$ and becomes very large even for moderate $q$. We now consider another construction from [13], which we call LPS-II, for which the number of vertices grows only linearly.

### C. LPS-II Graphs

Starting with two unequal primes $p$ and $q$, with $\left(\frac{p}{q}\right) = 1$, we form the Projective Line, $P^1(F_q)$, as

$$P^1(F_q) = \{0, 1, ..., q-1, \infty\} \tag{45}$$

which contains the field of integers modulo $q$, with an additional symbol $\infty$. The LPS-II graph is produced by the action of the set $S$ of $p + 1$ generators (considered in the LPS-I construction) on $P^1(F_q)$ in a linear fractional way. The graphs, thus obtained, are connected non-bipartite Ramanujan with degree $k = p+1$ and number of vertices $N = q+1$. Fig. 1 shows an LPS-II graph with $N = 62$ and $k = 6$ (obtained by choosing $p = 5$ and $q = 61$.)

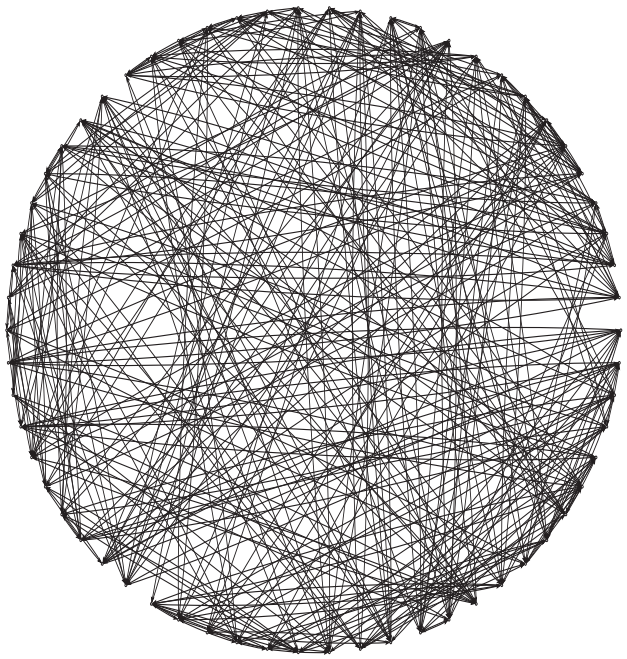

Fig. 1. LPS-II graph with number of vertices $N = 62$ and degree $k = 6$ (figure generated using software Pajek.)

Methods for the explicit construction of infinite families of Ramanujan graphs constitute an active research field and there are constructions for arbitrary $N$ and $k$, which are Ramanujan with very high probability. As an example see [20], which uses a new type of graph product to construct good expander graphs.

## IX. Numerical Studies

In this section we present numerical studies on the convergence properties of different networks, with respect to the distributed detection problem. In Section V, we have shown that the local probability of error, $P_e^n(i)$, at each sensor $n$, converges to the global probability of error, $P_e$. We define $T_c$ to be the number of iterations (averaged over all nodes) required to converge to within $10\%$ of the global probability of error. The comparison metric among different probabilities is then the convergence speed, $S_c$, given by

$$S_c = \frac{1}{T_c} \tag{46}$$

For a graph $G$ with $N$ vertices and $M$ edges, we define the average degree $k_{\text{avg}}$ as

$$k_{\text{avg}} = \frac{2M}{N} \tag{47}$$

It is to be noted that, for $k$-regular graphs, the average degree, $k_{\text{avg}} = k$. We now compare the convergence properties of the

non-bipartite Ramanujan graphs (in particular LPS-II graphs) with those of regular ring lattice networks, random Erdös-Renýi networks, and small world (Watts-Strogatz) networks. We first give a brief description of these graphs.

### A. Regular Ring Lattice (RRL)

These are highly structured $k$-regular graphs where the $N$ vertices are arranged on a circle, and each vertex is connected to its $k/2$ nearest neighbors on either side.

### B. Erdös-Renýi graphs (ER)

These are obtained by randomly choosing (uniformly) $M = \frac{Nk_{\text{avg}}}{2}$ edges out of $\frac{N(N-1)}{2}$ possible edges (see [21]).

### C. Watts-Strogatz (WS-I)

These are small-world networks. A WS-I graph with $N$ vertices and average degree $k_{\text{avg}}$ is constructed by randomly rewiring the edges of an RRL graph with $N$ vertices and degree $k_{\text{avg}}$, with a rewiring probability $0 \leq p_w \leq 1$ (see [22].) All the comparison studies are based on a detection environment with the following parameters (see Section V):

$$\mathbf{m} = \mu \mathbf{1}, \quad K = \sigma^2 I \tag{48}$$

with a $-25$ dB SNR (signal-to-noise ratio).

Fig. 2 compares the convergence properties of LPS-II graphs with RRL graphs. We plot the ratio of the convergence speeds, i.e., $\frac{S_{c(LPS-II)}}{S_{c(RRL)}}$, for different values of the number of vertices $N$, keeping the average degree $k_{\text{avg}} = 18$. It follows from the plot that the LPS-II graphs perform orders of magnitude better than the RRL graphs, and, in particular, the relative performance of the LPS-II graphs over the RRL graphs increases steadily with increasing $N$. In Fig. 3 we compare the LPS-II graphs with the Erdös-Renýi (ER) graphs. Here, also, we keep $k_{\text{avg}} = 18$ and vary the number of nodes. Since the ER constructions are random, for each $N$, we generate 250 different graphs, and plot the maximum, average, and minimum. We note that the LPS-II graphs perform much better than these random graphs, even for finite values of $N$, and the performance becomes better with increasing $N$. Finally, in Fig. 4 we compare the LPS-II graphs with WS-I small world graphs. Specifically, we fix the number of vertices at $N = 6038$ and average degree at $k_{\text{avg}} = 18$ and generate WS-I graphs for different rewiring probabilities $p_w$. Here, also, we note that the LPS-II graph performs better than the best WS-I graph (see [6] for more detailed numerical studies.)

These numerical studies show that the non-bipartite Ramanujan graphs (in particular, the LPS-II graphs) outperform topologies with nearest neighbor connectivity, completely random networks, and graphs with small-world type of connectivity, in terms of convergence speed. We see that the relation between $\gamma$ and the degree distribution, given in eqn.(41), plays an important role in explaining the superiority of Ramanujan graphs over the random networks. Specifically, it points out that graphs with heterogenous degree distributions (for example, Poisson, power-law tailed), cannot have very high values of $\gamma$.

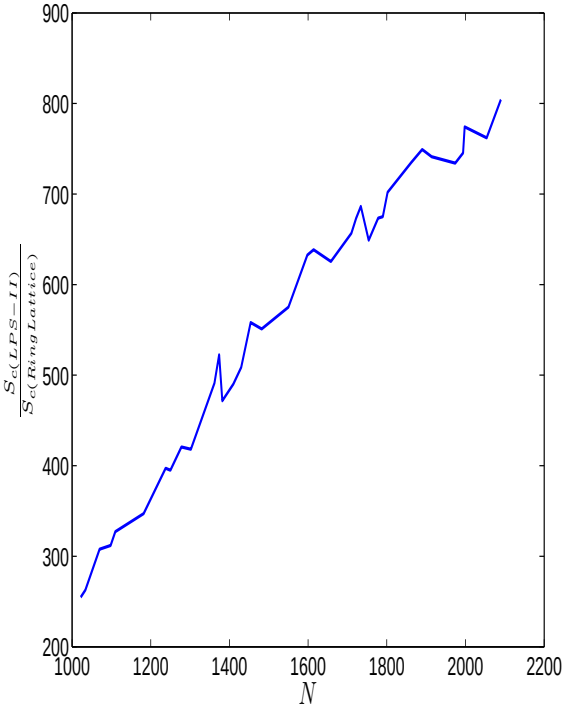


Fig. 2. Ratio of convergence speed of LPS-II graphs to RRL graphs for different values of $N$ and $k = 18$.

## X. Conclusions

In this paper, we study the problem of topology design for distributed detection in sensor networks. We reduce the connectivity network design problem to a spectral graph design problem, which shows that optimizing convergence speed is equivalent to maximizing the eigenratio $\gamma$. Using results from spectral graph theory, we show that the class of non-bipartite Ramanujan graphs are essentially optimal, in an asymptotic sense (in the number of sensors.) We also analytically establish the fact that a large heterogeneity in degree distribution does not favor good convergence rates. Hence, random networks with Poisson or power-law degree distributions are not optimal from the point of view of convergence speed. We supplement these facts through numerical studies, which show that, even for finite values of the number of sensors $N$, the non-bipartite Ramanujan graphs, outperform topologies with nearest neighbor connectivity, completely random networks, and networks with small-world properties.

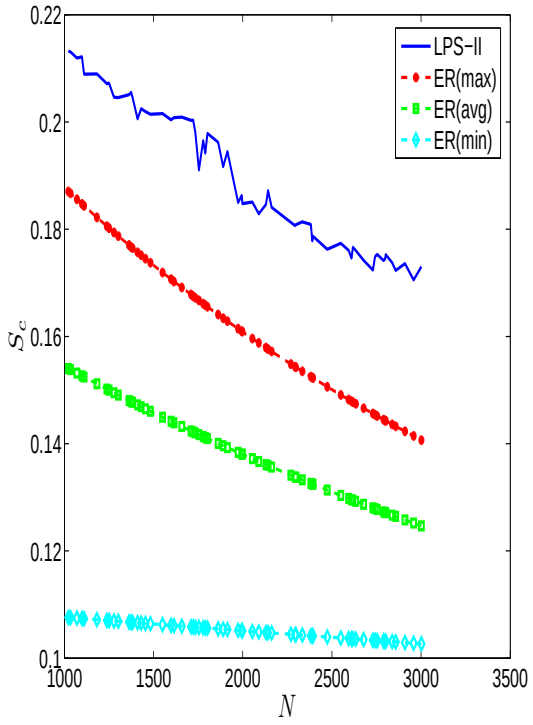


Fig. 3. Comparison of convergence speed between LPS-II graphs and Erdös-Renýi graphs for varying $N$ and $k_{\text{avg}} = 18$.

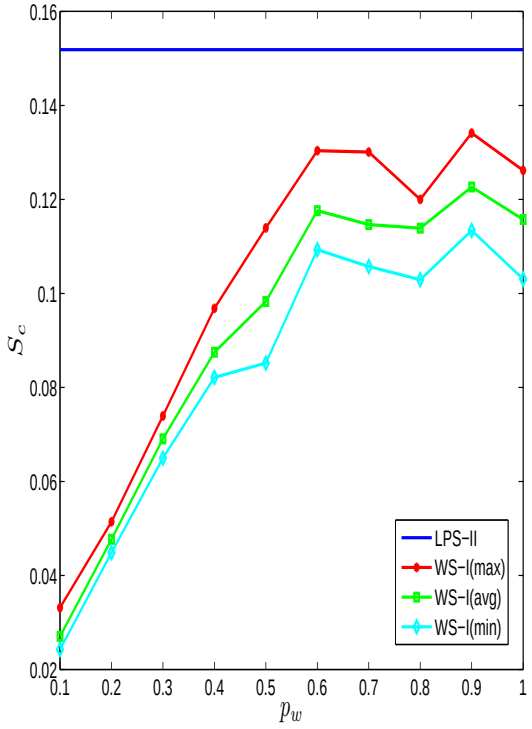


Fig. 4. Study of convergence speed $S_c$ of WS-I graph with $N = 6038$, $k_{\text{avg}} = 18$ for varying $p_w$. The top line denotes $S_c$ for the LPS-II graph with $N = 6038$ and $k = 18$.

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