First Order ODEs

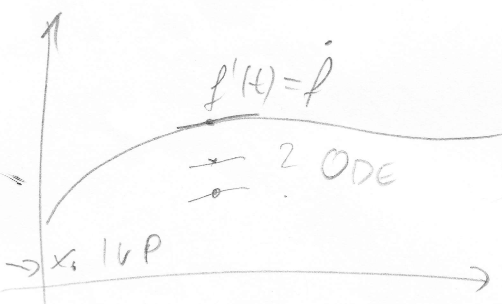
Scope

- Differential & Difference
- Ordinary, i.e. only one variable (usually time)
- constant coefficients
- Linear

Out of Scope

- PDEs (more than 1 variable)
- varying coefficients
- non linear

Differential equations: continuous function $f(t)$



$f(t) = \mathcal{F}(t, f, \dot{f})$

t

Difference equations: discrete time

$f(n) = \mathcal{F}(f(n-1), f(n), n)$

$$2\underbrace{\frac{dx(t)}{dt}}_{\text{1st order derivative}} + 4\underbrace{x(t)}_{\text{0th order derivative}} = \underbrace{6e^{-4t}}_{\text{forcing term}}, \quad \underbrace{t \geq 0}_{\text{time interval}}, \quad \underbrace{x(0) = 22}_{\text{initial condition}}$$

- t ... independent variable
- defined over $\mathbb{R}$
- ordinary
- 1st order
- time interval and initial cond.
- canonical form: $\frac{dx(t)}{dt}$, so divide out the constant

Solving an ODE

- function that satisfies the equation
- goes through initial value (IVP)

Questions: - existence } structural question
- Uniqueness

For first order ODES: Solutions exist, but not unique
IVPs: unique

1st order linear ODE with const coeff: Solution Procedure

Recipe: 4 steps: (given by general theory)

1) Homogenous solution
2) Particular solution
3) General solution
4) IVP solution

1) Homogenous solution

$$\frac{dx_h(t)}{dt} + 2x_h(t) = 0 \quad \forall t \geq 0$$

$|RHS \stackrel{!}{=} 0|$

$x_h \to$ homogeneous

guessing method (Ansatz):

$$x_h(t) = \alpha e^{\lambda t} \quad \forall t \geq 0$$

, α, λ parameters

$\uparrow$ natural freq
eigenvalue
mode
characteristic value

(we know the shape of the solution)

$\Rightarrow$ substitute in into ODE:

$$\alpha \lambda e^{\lambda t} + 2\alpha e^{\lambda t} = 0$$

$$\alpha e^{\lambda t}(\lambda + 2) = 0 \quad \Rightarrow \quad \alpha = 0 \text{ or } \lambda = -2$$

$$\Rightarrow x_h(t) = \alpha e^{-2t} \quad \forall t \geq 0$$

2) Particular solution

$x_p(t)$ is one solution that makes ODE an identity
- does not matter which one (seems strange)
- needed to get <u>all</u> solutions of ODE

$$\frac{dx_p(t)}{dt} + 2x_p(t) = -3e^{-4t}$$

<u>guessing of solution:</u>

$$x_p(t) = \beta e^{-4t} \qquad t \geq 0$$

$$-4\beta e^{-4t} + 2\beta e^{-4t} = -3e^{-4t}$$

$$\Rightarrow \beta = -\frac{3}{2}, \qquad x_p(t) = -\frac{3}{2}e^{-4t}, \quad t \geq 0$$

one should verify that $x_p(t)$ indeed is a solution to the ODE

3) General Solution

Theory: $x_g(t) = x_h(t) + x_p(t)$

$$\underline{x_g(t) = \alpha e^{-2t} - \frac{3}{2}e^{-4t}, \qquad t \geq 0}$$

check: substitute in LHS of ODE, derive RHS ✓

4) IVP

$$x_g(0) = \alpha e^{-2t}\Big|_{t=0} - \frac{3}{2} e^{-4t}\Big|_{t=0} = \alpha - \frac{3}{2} \stackrel{!}{=} 22$$

$$\Rightarrow \alpha = \frac{47}{2}$$

$$\Rightarrow x(t) = \frac{47}{2} e^{-2t} - \frac{3}{2} e^{-4t} \quad , \quad t \geq 0$$

Linearity w.r.t. initial condition

Consider an ODE / IVP with $f(t) \equiv 0$.

for initial condition

$$x_0 = \alpha_1 x_{01} + \alpha_2 x_{02}$$

the general solution is

$$x_g(t) = \alpha_1 x_{g1}(t) + \alpha_2 x_{g2}(t)$$

with $x_{g1}(t)$ general zero input solution for i.c. x_{01}

$x_{g2}(t)$ general zero input solution for i.c. x_{02}

Linearity w.r.t. forcing term (particular solution)

for forcing term

$$f(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t)$$

a particular solution is given by

$$x_p(t) = \alpha_1 x_{p_1}(t) + \alpha_2 x_{p_2}(t).$$

with

$x_{p_1}(t)$ is particular solution for $f_1(t)$

$x_{p_2}(t)$ is particular solution for $f_2(t)$

General Solution Approach

Always the 4 steps

- x_h
- x_p
- x_g
- $x_g(0) = x(0)$

Important ideas:

- Separation of variables
- Guessing Method
- Variation of constants
- linearity of x_p allows for var. const. solutions

Solve the Homogeneous: Separation of variables

$$\frac{dx(t)}{dt} = f(t), \quad x(0) = x_0$$

(special case)

"How to find x_h"

$$\int_0^t \frac{dx(t)}{dt}\,dt = \int_0^t f(t)\,dt$$

$$\int_{x_0}^{x(t)} dx(t) = \int_0^t f(t)\,dt$$

$$\Rightarrow \quad x(t) = x_0 + \int_0^t f(t)\,dt$$

Linearity

- both homogeneous and particular solutions have interesting linearity properties

Variation of Constants

How do find $x_p(t)$

Idea: $x_h(t)$ is already found (separation of variables)

Ansatz: $x_p(t) = c(t)\, x_h(t)$

$$\dot{x}_p(t) = \dot{x}_h(t)\, c(t) + x_h(t)\, \dot{c}(t)$$

substitute into ODE, simplify

$$\Rightarrow \quad x_h(t)\, \dot{c}(t) = f(t)$$

$$\Rightarrow \quad \frac{dc(t)}{dt} = \frac{f(t)}{x_h(t)}$$

separation of variables

$$c(t) = C_0 + \int_0^t x_h^{-1}(\tau) f(\tau)\, d\tau$$

$$x_p = x_h(t) \cdot c(t)$$

$\Rightarrow$ Closed form solution for $\dot{x} + a_0 x = f$, $x(0) = x_0$

$$x(t) = e^{-a_0 t} x_0 + \int_0^t e^{-a_0 (t-\tau)} f(\tau)\, d\tau \rightarrow \text{printout}$$

Stability ~~Behavior~~ at ∞ depends on $\lambda = -a_0$

$\operatorname{Re} \lambda < 0 \Rightarrow$ stable

$> 0 \rightarrow$ unstable

$= 0 \rightarrow$ bounded

Solution formula for IVP:

$$x(t) = e^{-a_0 t} x_0 + \int_0^t e^{-a_0 (t-\tau)} f(\tau)\, d\tau$$

Zero input response

for $f(t) \equiv 0$, $t \geq 0$

$x(t)$ reduces to $h_{zi}(t) = e^{-a_0 t} x_0$

Zero state response

for $x_0 = 0$ $x(t)$ reduces to

$$h_{zs}(t) = \int_0^t e^{-a_0 (t-\tau)} f(\tau)\, d\tau$$

Asymptotic Behavior

$$\lim_{x \to \infty} x(t)$$

Zero input response $x_{zi}(t)$, stability: sta

$$x_{zi}(t) = e^{-a_0 t} x_0$$

$$\lim_{t \to \infty} x_{zi}(t) = \lim_{t \to \infty} e^{-a_0 t} x_0 = \begin{cases} 0 & a_0 > 0 \\ x_0 & a_0 = 0 \\ \mathrm{sgn}(x_0) \cdot \infty & a_0 < 0 \end{cases}$$

unstable