Project 1 Part II: Metaproduct Primes

What you know
- Basic BDD data structure and JAVA implementation
- A little bit about these things called “metaproducts”

What you don’t know
- All the tricks with metaproducts
- Using these to do Prime Implicants

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About Metaproducts

Notation was created to support applications where we need to preserve the structure of things like SOP expressions

- ...ie, if you really WANT to write \( x + x' \)
- ...and you want to represent and manipulate it as a BDD, what to do?

Metaproduct notation

- Replace each variable “x” with a pair \((rx, sx)\)
- If you see \( x \) in a product, then you get \((rx)(sx)\) in metaproduct
- If you see \( x' \) in a product, then you get \((rx)(sx')\) in metaproduct
- If you don’t see any \( x \) or \( x' \) at all, then you get \((rx')\) in metaproduct

In English

- \( rx \) is the occurrence variable \( \rightarrow rx==1 \) says “\( x \) is here”, \( rx==0 \) says “no \( x \)”
- \( sx \) is the sign variable \( \rightarrow sx==1 \) says “\( x \) is positive”, \( sx==0 \) “negative”

Metaproduct Example

Suppose \( F(x,y) = x + xy' \)

- This is really just \( == x \), of course
- Its BDD would be simply

As a metaproduct

- Assume var ordering was \( x < y \)
- Then new ordering is \( x < rx < sx < y < ry < sy \)
- \( x \) becomes \((rx)(sx)(ry')(sy')\)
- \( xy' \) becomes \((rx)(sx)(ry)(sy')\)
Metaproduct Example

- You interpret this by looking at “satisfying paths” to “1” node
  - There are 2 paths from root to “1”, each makes a product term

\[
\begin{array}{cc}
0 & 1 \\
x & \text{x is here...} \\
\text{x is pos...} & y \\
\text{y is not here.}
\end{array}
\]

Result: \( x \)

\[
\begin{array}{cc}
0 & 1 \\
x & \text{x is here...} \\
\text{x is pos...} & y \\
\text{y is here...}
\end{array}
\]

Result: \( xy' \)

- Put the final products together for final answer: \( x + x'y \)

Metaproduct Example

- What happens if a variable is not present?
  - We already saw this, but its worth noting
  - In \( F(x,y) = x + xy' \), consider “x” term
  - There’s no “y” in there, but we still have to deal with “y”
  - Rule is: if there’s no variable in there, you still have to include the occurrence variables for the missing original vars, but you include them in the negative polarity to note their absence

\[
\text{So, term “x” becomes } (rx)(sx)(ty')
\]

Means “no y vars in here”

- If this was \( F(x,y,z,w) = x \), what would happen?
  - We’d get \( (rx)(sx)(ry')(rz')(rw') \)
Metaproduct Primes

Turns out you can represent Prime Implicants with metaproduct notation

xy zw

These are all Primes – biggest product terms you can circle in a Kmap for your function

xy zw

These are NOT all Primes – not all biggest product terms you can circle in this Kmap

Like with Boolean functions, problem is SIZE

- A function can have many many primes – way too many enumerate one by one
- This is why representing them with something like a BDD is attractive, since its good at “compressing” Boolean functions
- But this is also why we need a special notation, since we DON’T want the BDD to CHANGE our function to its canonical form
- We need to represent it in some SOP form

Big question: how do we find Primes using metaproducts

- Of course, it’s gotta be something recursive, right…?
Metaproduct Primes

There’s another Shannon-style recursive decomposition
- You start with a BDD in your original variables
- You end up with a BDD in the (occurrence, sign) metaproduct variables
- Final BDD represents, as SOP form, ALL the primes

Basic decomposition
- Let $P(BDD \text{ root of } F) = \text{metaproduct BDD for all PRIMES in function } F$

...why does that work?
- Roughly speaking –this is just the messy check for how to “reassemble” primes when they get split up during a Shannon decomposition

When we factored on $y$, we “chopped up” the prime like this

When we factored on $y$, we “chopped up” the prime like this
Metaproduct Primes

“Reading” the decomposition

\[ P(x) = P(L \ast H) \]

- If there’s no “x” vars, then get primes in \( L \ast H \)
- If neg “x’” var, get primes in \( L \) that are NOT in \( L \ast H \)
- If pos “x” var, get primes in \( L \) that are NOT in \( L \ast H \)

Let’s be a little more careful on the details of BDDs and ops

- Assumes you have AND and NOT (i.e., “!”) on BDDs
- Assumes \( P() \) calculated just like ITE, as a top-down recursion
- Assume var order is fixed: for varys \( x,y,... \), its: \( x < \_x \_x \_x < y < \_y \_y \_y \)

\[ P(x) = P(\text{AND}[P(), \neg P(\text{AND}[ , ]))] \]

\[ \text{AND}[P(), \neg P(\text{AND}[ , ]))] \]

\[ \text{AND}[P(), \neg P(\text{AND}[ , ]))] \]
Metaproduct Primes: Termination

So we know the recursion is:

\[ P(\ ) = \]

\[ P(\text{AND}[L,H]) \]

\[ \text{AND}[ P(0), \neg P(\text{AND}[L,H]) ] \]

\[ \text{AND}[ P(1), \neg P(\text{AND}[L,H]) ] \]

\[ x \]

...next question: what are the termination conditions for \( P(\ ) \)?

\[ \text{So, when can we quit, and return a known BDD node answer?} \]

\[ \text{Easy case: } P(0) = 0 \]

\[ \text{Harder case: } P(1) = \text{a little messy...} \]

Suppose vars are: \( x, y, z, w \), and we have this recursion

\[ P(\ ) = \]

\[ This \text{ just } ^{\text{“0”}} \]

\[ This \text{ is the product of complements of all occurrence vars later in the order, ie, after sx} \]

\[ In \text{ this case: } (ry')(rz')(rw') = \]

\[ 1 \]

\[ 0 \]

\[ 1 \]

Intuition

\[ \text{P(0) means “you're done – nothing more at all this prime term”} \]

\[ \text{P(1) means “you're done – but remember that these vars are absent”} \]
Primes Example

\[ f(x,y,z) = x' + y'z = 2 \text{ primes} \]

Apply recursion at root of \( f() \)

\[
\begin{align*}
P(\overline{x} ) &= P(x) \\
L &= H
\end{align*}
\]

Ordinary BDD with var order: \( x < y < z \)

Primes Example

Applying recursion at root of \( f() \)

\[
\begin{align*}
P(\overline{x} ) &= P(x) \\
\text{AND}[ P(), \overline{P()} ] \\
\text{AND}[ P(), \overline{P()} ] \\
\text{AND}[ P(), \overline{P()} ] \\
\end{align*}
\]
Primes Example

Do the obvious simplifications now (just to simplify for this manual example)

\[ P( ) = \]

\[ \text{AND} [ P( ), \neg P( ) ] \]

This was just
\[ P(\text{foo}) \oplus \neg P(\text{foo}) = 0 \]

Prime Example

\(<\text{OK, we need to do this one next}\>

\[ P( ) = \]

\[ \text{AND} [ P( ), \neg P( ) ] \]

\[ \text{AND} [ P( ), \neg P( ) ] \]

\[ \text{AND} [ P( ), \neg P( ) ] \]

\[ \text{AND} [ P( ), \neg P( ) ] \]

\[ \text{AND} [ P( ), \neg P( ) ] \]
Prime Example

Again, do obvious simplifications (just for this manual ex)

\[ P(0) = 0 \]
\[ P(1) = 1 \]
\[ P(0) \land P(1) = P(0) \land P(1) \]
\[ P(0) = 0 \]
\[ P(1) = 1 \]
\[ P(0) \land P(1) = 0 \]

We have to do this one next – and it's easy...

\[ P(0) = 0 \]
\[ P(1) = 1 \]
\[ P(0) \land P(1) = P(0) \land P(1) \]
\[ P(0) = 0 \]
\[ P(1) = 1 \]
\[ P(0) \land P(1) = 0 \]
Prime Example

We have to do this one next – and its easy...

\[ P(1) = \overline{z} \overline{1} \overline{0} \]

This is \( P(1) = \) product of complements of the vars later in the order than sz. Since sz is the LAST var in the order, the rule is: this is just “1”

Prime Example

Return results up recursive call tree...

Note – I'm leaving in all the separate “0” and “1” nodes just to simplify the drawing – it's a REAL BDD, there's only a single “1” and a single “0”…
Prime Example

\[ P(1) = \text{AND}[ P(0), \text{NOT}(1) ] \]

Since we know \( P(1) = \) product of complements of vars below \( sx \) in the order, this is supposed to be: \((ry')(rz')\), so we get...

\[ \text{AND}[ P(0), \text{NOT}(0) ] \]

...which is just ordinary BDD ops
Prime Example

BDD ops give this

\[ \text{AND} [ \ , \ \text{NOT}( ) ] \]

Return results up recursive call tree…

\[ P( ) = \]

\[ \text{AND} [ P( ), \text{NOT}( ) ] \]

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..and, that’s the Final Metaproduct for Prime( )

Look for paths from root to “1”

x is not here
y is here
y is negative
z is here
z is positive

This prime is \( y'z \)

Final Primes

More paths

x is here
x is negative
y is not here
...can ignore y sign
z is not here
...can ignore z sign

This prime is \( x' \)
Final Primes

\[ ... \text{hey, there's another one...?} \]

An unfortunate fact about metaproducts: they can be redundant about primes. You don't get the wrong ones – but you can get right ones several times.

This prime is also \( x' \)

Metaproduct Primes

\[ \text{So what did we do?} \]

Ordinary BDD with var order: \( x < y < z \)

BDD for Primes( \( f \) ) in metaproduct form

Primes = paths from root to “1”. Can be redundant.
What do we want?

- We want you to add $P(BDD \text{ for function } f)$ as an operator to your JAVA BDD package
- Do it exactly like we showed here
  - Just like ITE: you descend the starting BDD for $f$, and you recursively “trace out” the BDD for $P(f)$
  - Assume you have all the vars defined in the right initial order. This means if the real vars are $x, y$, YOU have $x, rx, sx, y, ry, sy$ in order
- You have 2 basic goals
  - To be able to transform a BDD for function $f$ into $Prime(f)$
  - To print out some interesting “info” about these primes

Things to be careful about

- Before doing anything, you probably want to build the function: $(rx')(ry')(rz')... (rlast')$ for ALL your vars. And make an array of pointers to the nodes, so that when you need $P(1) = \text{product of complemented occurrence nodes below me}$ – you can just look it up
- You still need to call FindOrCreateNode() on the 2 new nodes you make. You want to build sx and its children first, call FindOrCreateNode(sx), then finish the recursion on rx, then call FindOrCreateNode(sx).
- Do you want to do something like a different OPS table for the Prime computation? (It’s not required…but think about it)
- You will want to write a “printprime” routine that walks the paths to the “1” node, and prints out sensible product terms. DO NOT worry about the redundancy issue – not your problem.
- You also want to build a “numprimes” routine that just prints out the number of paths to the “1” node. Think about it – you don’t have to walk them all to do this, it’s a very simple recursion if you know numprimes(hichild) and numprimes(lochild), and numprimes(1)=1 and numprimes(0)=0
Metaproduct Primes: Summary

- Interesting, sort of funky BDD application
  - Twists the usual interpretation of "canonical BDD form" around a lot
  - Works fine, a bit arcane
    - (This is a simplification of how people really do it. There are a bunch of other optimizations to get rid of those redundancies that make it a lot faster. Not worth the grief to go thru them all...they violate a lot of BDD rules.)

- For Project 1
  - Implement Prime( f )
  - Look on the /afs/ece/class/ee760/proj1 directory for more details, and for some info about benchmarks to run
  - Ask TA and Prof questions if there are any issues at all on this one